Linear-Programming Applications

WEB CHAPTER PREVIEW  Most business resource-allocation problems require the decision maker to take into account various types of constraints, such as capital, labor, legal, and behavioral restrictions. Linear-programming techniques can be used to provide relatively simple and realistic solutions to problems involving constrained resource-allocation decisions. A wide variety of production, finance, marketing, and distribution problems have been formulated in the linear-programming framework. Consequently, managers should understand the linear-programming model so they may allocate the resources of the enterprise most efficiently, particularly in situations where important constraints are placed on the actions that may be taken. The chapter begins by developing the formulation and graphical solution to a profit-maximization production problem. The following section discusses the concept of dual variables and their interpretations. A computer solution to a cost-minimization problem is presented next. Finally, the formulation and solution of two problems from finance and distribution are presented.

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A PROFIT-MAXIMIZATION PROBLEM

This section discusses the formulation of linear-programming problems and presents a graphical solution to a simple profit-maximization problem.

Statement of the Problem

A multiproduct firm often has the problem of determining the optimal product mix, that is, the combination of outputs that will maximize its profits. The firm is normally subject to various constraints on the amount of resources, such as raw materials, labor, and production capacity, that may be employed in the production process.

Example

PROFIT MAXIMIZATION: WHITE COMPANY

Consider the White Company, a manufacturer of gas (Product 1) and electric (Product 2) clothes dryers. The problem is to determine the optimal level of output ($X_1$ and $X_2$) for two products (1 and 2). Information about the problem is summarized in Table B.1. Production consists of a machining process that takes raw materials and converts them into unassembled parts. These are then sent to one of two divisions for assembly into the final product—Division 1 for Product 1 and Division 2 for Product 2.\(^3\) As listed in Table B.1, to service the aircraft. Furthermore, schedulers must be able to deal with disruptions caused by bad weather and emergency changes in shipping priorities. Adding just a couple of percentage points to the efficiency of the airlift system can save the Air Force millions of dollars annually in equipment, labor, and fuel costs. Major commercial airlines, such as American and United, face similar scheduling problems. Complex resource-allocation problems such as these can be solved using linear-programming techniques.


\(^3\) This problem ignores any scheduling difficulties that may exist in the production process.

MANAGERIAL CHALLENGE

Military Airlift Command\(^2\)

The United States Air Force’s Military Airlift Command (MAC) uses approximately 1,000 planes (of varying capacity, speed, and range) to ferry cargo and passengers among more than 300 airports scattered around the world. Resource constraints, such as the availability of planes, pilots, and other flight personnel, place limitations or constraints on the capacity of the airlift system. Additionally, MAC must determine whether it is more efficient to reduce cargo and top off the fuel tanks at the start of each flight or to refuel at stops along the way and pay for the costs of shipping fuel. The airlift system also requires that cargo handlers and ground crews be available to service the aircraft. Furthermore, schedulers must be able to deal with disruptions caused by bad weather and emergency changes in shipping priorities. Adding just a couple of percentage points to the efficiency of the airlift system can save the Air Force millions of dollars annually in equipment, labor, and fuel costs. Major commercial airlines, such as American and United, face similar scheduling problems. Complex resource-allocation problems such as these can be solved using linear-programming techniques.

Product 1 requires 20 units of raw material and 5 hours of machine-processing time, whereas Product 2 requires 40 units of raw material and 2 hours of machine-processing time. During the period, 400 units of raw material and 40 hours of machine-processing time are available. The capacities of the two assembly divisions during the period are 6 and 9 units, respectively. The operating profit contribution per unit or, more accurately, the per-unit contribution to profit and overhead (fixed costs) is $100 for each unit of Product 1 and $60 for each unit of Product 2. The contribution per unit represents the difference between the selling price per unit and the variable cost per unit. With this information, the problem can be formulated in the linear-programming framework.

### Formulation of the Linear-Programming Problem

#### Objective Function
The objective is to maximize the total contributions $\pi$ from the production of the two products, where total profit contribution is equal to the sum of the contribution per unit of each product times the number of units produced. Therefore, the objective function is

$$\text{Max } \pi = 100X_1 + 60X_2 \quad \text{[B.1]}$$

where $X_1$ and $X_2$ are, as defined earlier, the output levels of Products 1 and 2, respectively.

#### Constraint Relationships
The production process described has several resource constraints imposed on it. These need to be incorporated into the formulation of the problem. Consider first the raw material constraint. Production of $X_1$ units of Product 1 requires $20X_1$ units of raw materials. Similarly, production of $X_2$ units of Product 2 requires $40X_2$ units of the same raw material. The sum of these two quantities of raw materials must be less than or equal to the quantity available, which is 400 units. This relationship can be expressed as

$$20X_1 + 40X_2 \leq 400 \quad \text{[B.2]}$$

The machine-processing time constraint can be developed in a like manner. Product 1 requires $5X_1$ hours and Product 2 requires $2X_2$ hours. With 40 hours of processing time available, the following constraint is obtained:

$$5X_1 + 2X_2 \leq 40 \quad \text{[B.3]}$$
The capacities of the two assembly divisions also limit output and consequently profits. For Product 1, which must be assembled in Division 1, the constraint is

\[ X_1 \leq 6 \quad [B.4] \]

For Product 2, which must be assembled in Division 2, the constraint is

\[ X_2 \leq 9 \quad [B.5] \]

Finally, the logic of the production process suggests that negative output quantities are not possible. Therefore, each of the decision variables is constrained to be nonnegative:

\[ X_1 \geq 0 \quad X_2 \geq 0 \quad [B.6] \]

Equations B.1 through B.6 constitute a linear-programming formulation of the profit-maximization production problem.

**Economic Assumptions of the Linear-Programming Model**

In formulating this problem as a linear-programming model, one must understand the economic assumptions that are incorporated into the model. Basically, one assumes that a series of linear (or approximately linear) relationships involving the decision variables exist over the range of alternatives being considered in the problem. For the resource inputs, one assumes that the *prices of these resources to the firm are constant* over the range of resource quantities under consideration. This assumption implies that the firm can buy as much or as little of these resources as it needs without affecting the per unit cost.\(^4\) Such an assumption would rule out quantity discounts. One also assumes that there are *constant returns to scale* in the production process. In other words, in the production process, a doubling of the quantity of resources employed doubles the quantity of output obtained, for any level of resources.\(^5\) Finally, one assumes that the *market selling prices of the two products are constant* over the range of possible output combinations.\(^6\) These assumptions are implied by the fixed per-unit profit contribution coefficients in the objective function. If the assumptions are not valid, then the optimal solution to the linear-programming model will not necessarily be an optimal solution to the actual decision-making problem. Although these relationships need not be linear over the entire range of values of the decision variables, the linearity assumptions must be valid over the full range of values being considered in the problem.

**Graphical Solution of the Linear-Programming Problem**

Various techniques are available for solving linear-programming problems. For larger problems involving more than two decision variables, one needs to employ algebraic methods to obtain a solution. Further discussion of these methods is postponed until later in the chapter. For problems containing only two decision variables, graphical methods can be used to obtain an optimal solution. To understand the nature of the objective function and constraint relationships, it is helpful to solve the preceding prob-

\(^4\) This assumption involves the concept of an atomistic buyer in a competitive factor or input market. See Chapter 10 for a discussion of this type of market.

\(^5\) “Doubling the quantity of resources” is used as an example. More generally, one would say that a given percentage increase in each of the resources would result in an equivalent percentage increase in output for any given level of resources. See Chapter 7 for a further discussion of the concept of returns to scale.

\(^6\) This assumption is satisfied in a perfectly competitive market for the two final products. Further discussion of this type of market is in Chapter 10.
lem graphically. This is done by graphing the feasible solution space and objective function separately and then combining the two graphs to obtain the optimal solution.

**Profit Maximization: White Company (continued)**

*Graphing the Feasible Solution Space*  
Note from Equation B.6 that each of the decision variables must be greater than or equal to zero. Therefore, one needs only graph the upper right-hand (positive) quadrant. Figure B.1 illustrates the raw material constraint as given by Equation B.2. The upper limit or maximum quantity of raw materials that may be used occurs when the inequality is satisfied as an equality; in other words, the set of points that satisfies the equation

\[ 20X_1 + 40X_2 = 400 \]

Because it is possible to use less than the amount of raw materials available, any combination of outputs lying on or below this line (that is, the shaded area) will satisfy the raw materials constraint.

Similarly, the constraint on the amount of machine-processing time (in hours) available (Equation B.3) yields the combinations of \( X_1 \) and \( X_2 \) that lie on or below the line (that is, the shaded area) shown in Figure B.2. Likewise, one can determine the set of feasible combinations of \( X_1 \) and \( X_2 \) for each of the remaining constraints (Equations B.4 and B.5).

Combining all the constraints (Equations B.2–B.6) yields the feasible solution space (shaded area) shown in Figure B.3, which simultaneously satisfies all the constraints of

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**Feasible Solution Space**  
The set of all possible combinations of the decision variables that simultaneously satisfies all the constraints of the problem.
the problem. All possible production combinations of $X_1$ and $X_2$ that simultaneously satisfy all the resource constraints lie in or on the boundary of the shaded area.

**Graphing the Objective Function** The objective function given by Equation B.1 specifies the profit that will be obtained from any combination of output levels. The profit function can be represented graphically as a series of parallel *isoprofit* lines. Each of the lines shown in Figure B.4 is an isoprofit line, meaning that each combination of output levels (that is, $X_1$ and $X_2$) lying on a given line has the *same* total profit. For example, the $\pi = 1,200$ isoprofit line includes such output combinations as $(X_1 = 6, X_2 = 10)$ and $(X_1 = 9, X_2 = 5)$. The objective of profit maximization can be interpreted graphically to find an output combination that falls on as high an isoprofit line as possible. The resource constraints of the problem obviously limit us from increasing output and profits indefinitely.

**Graphical Solution** Combining the graphs of the feasible solution space and objective function yields the output combination point within the feasible solution space that lies on the highest possible isoprofit line. The two graphs have been combined in Figure B.5. From the graph it can be seen that the optimal output combination at point $C$ is $X_1^* = 5$ units and $X_2^* = 7.5$ units, yielding a profit of

$$\pi^* = 100 \times 5 + 60 \times 7.5 = 950$$

No other output combination within the feasible solution space will result in a larger profit.
**FIGURE B.3**
Feasible Solution Space: Profit-Maximization Problem

Machine-processing time constraint
\[ 5X_1 + 2X_2 \leq 40 \]

Division 1 capacity constraint
\[ X_1 \leq 6 \]

Division 2 capacity constraint
\[ X_2 \leq 9 \]

Raw material constraint
\[ 20X_1 + 40X_2 \leq 400 \]

**FIGURE B.4**
Isoprofit Lines: Profit-Maximization Problem

\[ \pi = 100X_1 + 60X_2 \]
Sometimes it is difficult to read the exact coordinates of the optimal solution from the graph. When this occurs (or when one wants to confirm the solution algebraically), we can determine the exact solution by solving simultaneously the equations of the two lines passing through the optimum point. In the preceding example, the equations of the two lines passing through point \( C \) are

\[
20X_1 + 40X_2 = 400 \\
5X_1 + 2X_2 = 40
\]

which correspond to the raw materials and machine-processing time constraints, respectively. Solving these two equations simultaneously does indeed yield \( X_1^* = 5 \) and \( X_2^* = 7.5 \)—the same result that was obtained graphically.

**Extreme Points and the Optimal Solution**

This example demonstrates two important general properties of an optimal solution to a linear-programming problem. These properties are useful in developing algebraic solutions to this class of problem, and they form the foundation for computer (algorithmic) solution techniques. First, note that the optimal solution lies on the boundary of the feasible solution space. The implication of this property is that one can ignore the infinite number of interior points in the feasible solution space when searching for an optimal solution. Second, note that the optimal solution occurs at one of the extreme points...
points (corner points) of the feasible solution space. This property reduces even further the magnitude of the search procedure for an optimal solution. For this example it means that from among the infinite number of points lying on the boundary of the feasible solution space, only six points—A, B, C, D, E, and zero—need to be examined to find an optimal solution.

**Multiple Optimal Solutions**

A problem also can have *multiple* optimal solutions if the isoprofit line coincides with one of the boundaries of the feasible solution space. For example, if the objective function in the production problem were equal to

\[
\pi' = 100X_1 + 40X_2
\]  
[B.7]

then the isoprofit line \( \pi' = 800 \) would coincide with the CD boundary line of the feasible solution space as illustrated in Figure B.6. In this case both corner points C and D, along with all the output combinations falling along the CD line segment, would constitute optimal solutions to the problem.

**Slack Variables**

In addition to the optimal combination of output to produce \( X_1^* \) and \( X_2^* \) and the maximum total profit \( \pi^* \), we are also interested in the amount of each resource used in the production process. For the production of 5 units \( \times 1 \) and 7.5 units \( \times 2 \) of...
products 1 and 2, respectively, the resource requirements (from Equations B.2–B.5) are as follows:

\[
\begin{align*}
20(5) + 40(7.5) &= 400 \text{ units of raw materials} \\
5(5) + 2(7.5) &= 40 \text{ hours of machine-processing time} \\
1(5) &= 5 \text{ units of Division 1 assembly capacity} \\
1(7.5) &= 7.5 \text{ units of Division 2 assembly capacity}
\end{align*}
\]

This information indicates that all available raw materials (400 units) and all available machine-processing time (40 hours) will be used in producing the optimal output combination. However, 1 unit of Division 1 assembly capacity (6) and 1.5 units of Division 2 assembly capacity (9) will be unused in producing the optimal output combination. These unused or idle resources associated with a less than or equal to constraint (\(\leq\)) are referred to as slack.

**Slack variables** can be added to the formulation of a linear-programming problem to represent this slack or idle capacity. Slack variables are given a coefficient of zero in the objective function because they make no contribution to profit. Slack variables can be thought of as representing the difference between the right-hand side and left-hand side of a less than or equal to inequality (\(\leq\)) constraint.

In the preceding profit-maximization problem (Equations B.1–B.6), four slack variables \(S_1, S_2, S_3, S_4\) are used to convert the four (less than or equal to) constraints to equalities as follows:

\[
\begin{align*}
\text{Max } \pi &= 100X_1 + 60X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 \\
20X_1 + 40X_2 + 1S_1 &= 400 \\
5X_1 + 2X_2 + 1S_2 &= 40 \\
X_1 + 1S_3 &= 6 \\
X_2 + 1S_4 &= 9 \\
X_1, X_2, S_1, S_2, S_3, S_4 &\geq 0
\end{align*}
\]

As shown later in the chapter, a computer solution of a linear-programming problem automatically provides the optimal values of the slack variables along with the optimal values for the original decision variables.

**The Dual Problem and Interpretation of the Dual Variables**

The solution of a linear-programming problem, in addition to providing the optimal values of the decision variables, contains information that can be very useful in making marginal resource-allocation decisions. This marginal information is contained in what are known as the **dual variables** of the linear-programming problem.

**The Dual Linear-Programming Problem**

Associated with every linear-programming problem is a related **dual** linear-programming problem. The *originally formulated* problem, in relation to the dual problem, is known as the **primal** linear-programming problem. If the objective in the primal problem is maximization of some function, then the objective in the dual problem is minimization of a related (but different) function. Conversely, a primal minimization problem has a related dual maximization problem. The dual variables represent the variables contained in the dual problem.

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*By symmetry, the dual of the dual problem is the primal problem.*
PROFIT MAXIMIZATION: WHITE COMPANY (CONTINUED)

Before indicating how the dual variables can be used as an aid in marginal decision-making, it may be useful to illustrate, using the White Company profit-maximization problem, the relation of the dual problem to the primal problem. One way to show the relationship is by means of a matrix diagram, such as the one in Figure B.7. The primal problem is contained in the rows of the figure. For example, the first row (W₁) of numbers in the figure contains the Equation B.2 constraint; that is, 20X₁ + 40X₂ ≤ 400. The last row (Constants) contains the objective function (Equation B.1); that is, max π = 100X₁ + 60X₂.

Associated with each constraint of the primal problem is a dual variable. Because the primal problem had four constraints, the dual problem has four variables—W₁, W₂, W₃, and W₄. The dual problem is contained in the columns of the figure. The objective of the dual problem is contained in the Constants column:

\[
\text{Min } Z = 400W₁ + 40W₂ + 6W₃ + 9W₄ \tag{B.8}
\]

Similarly, the constraints of the dual problem are contained under the X₁ and X₂ columns:

\[
\begin{align*}
20W₁ + 5W₂ + W₃ & \geq 100 \tag{B.9} \\
40W₁ + 2W₂ + W₄ & \geq 60 \tag{B.10}
\end{align*}
\]

One also requires

\[
W₁ \geq 0, \ W₂ \geq 0, \ W₃ \geq 0, \ W₄ \geq 0 \tag{B.11}
\]

In general, a primal problem with \( n \) variables and \( m \) constraints will have as its dual a problem with \( m \) variables and \( n \) constraints.

Economic Interpretation of the Dual Variables

In the preceding resource-constrained profit-maximization problem, a dual variable existed for each of the limited resources required in the production process. In such a problem, the dual variables measure the “imputed values” or shadow prices of each of the scarce resources. Expressed in dollars per unit of resource, they give an indication of...
how much each resource contributes to the overall profit function. With this interpretation of the dual variables, the dual objective function (Equation B.8) is to minimize the total cost or value of the resources employed in the process. The two dual constraints (Equations B.9 and B.10) require that the value of the resources used in producing one unit each of \( X_1 \) and \( X_2 \) be at least as great as the profit received from the sale of one unit of each product. An important linear-programming theorem, known as the duality theorem, indicates that the maximum value of the primal profit function will also be equal to the minimum value of the dual “imputed value” function.\(^9\) The solution of the dual problem in effect apportions the total profit figure among the various scarce resources employed in the process.

The interpretation of the dual variables and dual problem depends on the nature and objective of the primal problem. Thus a completely different interpretation is involved whenever the primal problem is one of cost minimization.\(^10\)

### Profit Maximization: White Company (continued)

The preceding example illustrates how the dual variables can be used to make marginal resource-allocation decisions. The values of the dual variables, which are obtained automatically in an algebraic solution of the linear-programming problem, are \( W_1 = 0.625 \) per unit, \( W_2 = 17.50 \) per unit, \( W_3 = 0 \) per unit, and \( W_4 = 0 \) per unit. Each dual variable indicates the rate of change in total profits for an incremental change in the amount of each of the various resources. In this way they are similar to the \( \lambda \) values used in the Lagrangian multiplier technique. The dual variables indicate how much the total profit will change (i.e., marginal profit) if one additional unit of a given resource is made available, provided the increase in the resource does not shift the optimal solution to another corner point of the feasible solution space. For example, \( W_2 = 17.50 \) indicates that profits could be increased by as much as $17.50 if an additional unit (hour) of machine capacity could be made available to the production process. This type of information is potentially useful in making decisions about purchasing or renting additional machine capacity or using existing machine capacity more fully through the use of overtime and multiple shifts. A dual variable equal to zero, such as \( W_3 \) and \( W_4 \), indicates that profits would not increase if additional resources of these types were made available; in fact, excess capacity in these resources exists. (Recall in the discussion of slack variables, portions of these resources were unused or idle in the optimal solution.) This discussion only indicates the type of analysis that is possible. Much more detailed analysis of this nature can be performed using parametric-programming techniques.\(^11\)

### A Cost-Minimization Problem

This section develops a cost-minimization problem and illustrates the use of computer programs for its solution.

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**Statement of the Problem**

Large multiplant firms often produce the same products at two or more factories. Often these factories employ different production technologies and have different unit production costs. The objective is to produce the desired amount of output using the given facilities (that is, plants and production processes) to minimize production costs.

**COST MINIMIZATION: SILVERADO MINING COMPANY**

Suppose that the Silverado Mining Company owns two different mines (A and B) for producing uranium ore. The two mines are located in different areas and produce different qualities of uranium ore. After the ore is mined, it is separated into three grades—high-, medium-, and low-grade. Information concerning the operation of the two mines is shown in Table B.2. Mine A produces .75 tons of high-grade ore, .25 tons of medium-grade ore, and .50 tons of low-grade ore per hour. Likewise, Mine B produces .25, .25, and 1.50 tons of high-, medium-, and low-grade ore per hour, respectively. The firm has contracts with uranium-processing plants to supply a minimum of 36 tons of high-grade ore, 24 tons of medium-grade ore, and 72 tons of low-grade ore per week. These figures are shown in the Requirements column of Table B.2. Finally, as shown in the bottom row of Table B.2, it costs the company $50 per hour to operate Mine A and $40 per hour to operate Mine B. The company wishes to determine the number of hours per week it should operate each mine to minimize the total cost of fulfilling its supply contracts.

**Formulation of the Linear-Programming Problem**

**Objective Function** The objective is to minimize the total cost per week \(C\) from the operation of the two mines, where the total cost is equal to the sum of the operating cost per hour of each mine times the number of hours per week that each mine is operated. Defining \(X_1\) as the number of hours per week that Mine A is operated and \(X_2\) as the number of hours per week that Mine B is operated, the objective function is

\[
\text{Min } C = 50X_1 + 40X_2 \quad [B.12]
\]

**Constraint Relationships** The Silverado Mining Company’s contracts with uranium-processing plants require it to operate the two mines for a sufficient number of hours to produce the required amount of each grade of uranium ore. In the production of

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**Example**

<table>
<thead>
<tr>
<th>Mine</th>
<th>Output (Tons of Ore Per Hour)</th>
<th>Requirements (Tons Per Week)</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-grade ore</td>
<td>.75</td>
<td>.25</td>
</tr>
<tr>
<td>Medium-grade ore</td>
<td>.25</td>
<td>.25</td>
</tr>
<tr>
<td>Low-grade ore</td>
<td>.50</td>
<td>1.50</td>
</tr>
<tr>
<td>Mine</td>
<td>Operating cost ($/hour)</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>
high-grade ore, Mine A produces .75 tons per hour times the number of hours per week \( (X_1) \) that it operates, and Mine B produces .25 tons per hour times the number of hours per week \( (X_2) \) that it operates. The sum of these two quantities must be greater than or equal to the required output of 36 tons per week. This relationship can be expressed as

\[
.75X_1 + .25X_2 \geq 36 \quad \text{[B.13]}
\]

Similar constraints can be developed for the production of medium-grade ore

\[
.25X_1 + .25X_2 \geq 24 \quad \text{[B.14]}
\]

and low-grade ore

\[
.50X_1 + 1.50X_2 \geq 72 \quad \text{[B.15]}
\]

Finally, negative production times are not possible. Therefore, each of the decision variables is constrained to be nonnegative:

\[
X_1 \geq 0, \quad X_2 \geq 0 \quad \text{[B.16]}
\]

Equations B.12 through B.16 represent a linear-programming formulation of the cost-minimization production problem.

**Slack (Surplus) Variables**

Recall from the discussion of the maximization problem earlier in the chapter that slack variables were added to the less than or equal to inequality \((\leq)\) constraints to convert these constraints to equalities. Similarly, in a minimization problem, surplus variables are subtracted from the greater than or equal to inequality \((\geq)\) constraints to convert these constraints to equalities. Like the slack variables, surplus variables are given coefficients of zero in the objective function because they have no effect on the value.

**Cost Minimization: Silverado Mining Company (continued)**

In the preceding cost-minimization problem, three surplus variables \((S_1, S_2, S_3)\) are used to convert the three (greater than or equal to) constraints to equalities as follows:

\[
\text{Min } C = 50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3
\]

s.t.

\[
.75X_1 + .25X_2 - 1S_1 = 36
\]

\[
.25X_1 + .25X_2 - 1S_2 = 24
\]

\[
.50X_1 + 1.50X_2 - 1S_3 = 72
\]

\[
X_1, X_2, S_1, S_2, S_3 \geq 0
\]

**Computer Solution of the Linear-Programming Problem**

The solution of large-scale linear-programming problems typically employs a procedure (or variation of the procedure) known as the simplex method. Basically, the simplex method is a step-by-step procedure for moving from corner point to corner point of the feasible solution space in such a manner that successively larger (or smaller) values of the maximization (or minimization) objective function are obtained at each step. The procedure is guaranteed to yield the optimal solution in a finite number of steps. Further discussion of this method is beyond the scope of this chapter.\(^\text{12}\)

\(^\text{12}\)Any basic linear-programming textbook, such as the previously cited Anderson et al., Dantzig, and Hadley books, contains detailed discussions of this procedure.
Most practical applications of linear programming use computer programs to perform the calculations and obtain the optimal solution. Although many different programs are available for solving linear-programming problems, the output of these programs usually includes the optimal solution to the primal problem as well as the optimal values of the dual variables. The particular program illustrated here is known as SIMPLX.13 (Similar programs are likely to be readily available on your personal computer or school’s computer system.)

COST MINIMIZATION: SILVERADO MINING COMPANY (CONTINUED)

Putting the objective function and constraints (Equations B.12 through B.16) along with the appropriate control statements into the SIMPLX program yields the output shown in Figure B.8. The optimal values of the decision variables are shown in the Primal So-

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13“SIMPLX” is a terminal-oriented computer program. See E. Pearsall and B. Price, Linear Programming and Simulation (No. MS(350)), CONDUIT (Ames: Iowa State University).
Problem 1.105

The ore-mining firm should operate Mine A for 24 hours per week and Mine B for 72 hours per week to minimize total operating costs. This yields a minimum total cost of $4,080 per week.

Note also that the optimal value of the surplus variable $S_1$ [that is, $X_1^*$ on the computer output] is 48. This indicates that a surplus of 48 tons of low-grade ore (that is, 120 tons versus the required amount of 72 tons) is being produced in the optimal solution. Similarly, $S_1$ and $S_2$ are zero (all variables not listed in the primal solution are equal to zero), indicating that exactly the required amounts of high-grade and medium-grade ore (36 and 24 tons, respectively) are being produced in the optimal solution.

Recall from the earlier discussion of the dual problem and dual variables that a dual variable is associated with each constraint equation in the primal problem (excluding non-negativity constraints). The ore-mining problem has three constraint equations—one for each of the three types of uranium ore. Consequently, it has three dual variables—$W_1$, $W_2$, and $W_3$—associated with each of the respective constraint equations. The optimal values of the dual variables are shown in the Dual Solution column of Figure B.8—$W_1^* = 20$, $W_2^* = 140$, and $W_3^* = 0$. Each dual variable measures the change in total cost (i.e., marginal cost) that results from a one-unit (ton) increase in the required output, provided that the increase does not shift the optimal solution to another corner point of the feasible solution space. For example, $W_1^* = 20$ indicates that total costs will increase by as much as $20 if the firm is required to produce an additional ton of high-grade uranium ore. Comparison of this value to the revenue received per ton of ore can help the firm in making decisions about whether to expand or contract its mining operations.

Next, consider the interpretation of $W_3^* = 0$. This zero value indicates that surplus low-grade ore is being produced by the firm. (Recall that $S_3^* = 48$.) At the optimal solution (operating Mines A and B at 24 and 72 hours per week, respectively), the cost of producing an additional ton of low-grade ore is $0.

A NEW TECHNIQUE FOR SOLVING LARGE-SCALE LINEAR-PROGRAMMING PROBLEMS

Since its development in 1947 by operations research pioneer George Dantzig, most linear-programming problems have been solved using the simplex method (or variations thereof). Approximately 80 to 90 percent of these constrained optimization problems can be solved on computers using this algorithm. However, when solving extremely large problems or problems that are changing rapidly, the simplex method often is too slow to be practical.

An AT&T Bell Laboratories researcher, Narendra Karmarkar, developed an alternative solution technique that is potentially 50 to 100 times faster than the simplex method in solving large, complex linear-programming problems. For example, Bell Laboratories (now Lucent Technologies) is using Karmarkar’s algorithm to forecast the most cost-effective way to satisfy the future needs over a 10-year horizon of the telephone network linking 20 countries on the rim of the Pacific Ocean. The resulting linear-programming problem contains 42,000 variables. Solving this problem using the simplex method would require 4 to 7 hours of mainframe computer time to answer each “what-if” question, whereas this new technique would require less than 4 minutes.

In another application, Karmarkar’s algorithm was used to solve the Military Airlift Command’s scheduling problem described in the Managerial Challenge section at the beginning of the chapter. Solving this linear-programming problem, which involves 321,000 variables and 14,000 constraints, required only one hour of computer time—just a fraction of the time that would be required using the simplex method. Given the new method’s ability to solve large problems quickly, the Military Airlift Command, as
well as commercial airlines such as American and United, should be able to solve complex scheduling problems and make efficient adjustments rapidly in response to changing operating constraints.

**ADDITIONAL LINEAR-PROGRAMMING EXAMPLES**

Linear programming is useful in a wide variety of managerial resource-allocation problems. This section examines some additional applications in finance, marketing, and distribution.

**THE CAPITAL-RATIONING PROBLEM: ASPEN SKI COMPANY**

Rather than letting the size of their capital budgets (expenditures that are expected to provide long-term benefits to the firm, such as plants and equipment) be determined by the number of profitable investment opportunities available (all investment projects meeting some acceptance standard), many firms place an upper limit or constraint on the amount of funds allocated to capital investment. **Capital rationing** takes place whenever the total cash outlays for all projects that meet some acceptance standard exceed the constraint on total capital investment.

For example, suppose that the Aspen Ski Company is faced with the set of nine investment projects shown in Table B.3, requiring the outlay of funds in each of the next two years (shown in columns 2 and 3) and generating the returns (net present values) shown in column 4.\(^{14}\) Furthermore, suppose the firm has decided to limit total capital expenditures to $50,000 and $20,000 in each of the next two years, respectively. The problem is to select the combination of investments that provides the **largest possible return** (net present value) without violating either of the two constraints on total capital expenditures. This problem can be formulated and solved using linear-programming techniques.

---

Begin by defining $X_j$ to be the fraction of project $j$ undertaken (where $j = 1, 2, 3, 4, 5, 6, 7, 8$, and $9$). The objective is to maximize the sum of the returns (net present value) of the projects undertaken:

$$\text{Max } R = 14X_1 + 17X_2 + 17X_3 + 15X_4 + 40X_5 + 12X_6 + 14X_7 + 10X_8 + 12X_9 \quad [B.17]$$

The constraints are the restrictions placed on total capital expenditures in each of the two years:

$$12X_1 + 54X_2 + 6X_3 + 6X_4 + 30X_5 + 6X_6 + 48X_7 + 36X_8 + 18X_9 \leq 50 \quad [B.18]$$

$$3X_1 + 7X_2 + 6X_3 + 2X_4 + 35X_5 + 6X_6 + 4X_7 + 3X_8 + 3X_9 \leq 20 \quad [B.19]$$

Also, so that no more than one of any project will be included in the final solution, all the $X_j$’s must be less than or equal to 1:

$$X_1 \leq 1 \quad [B.20]$$
$$X_2 \leq 1 \quad [B.21]$$
$$X_3 \leq 1 \quad [B.22]$$
$$X_4 \leq 1 \quad [B.23]$$
$$X_5 \leq 1 \quad [B.24]$$
$$X_6 \leq 1 \quad [B.25]$$
$$X_7 \leq 1 \quad [B.26]$$
$$X_8 \leq 1 \quad [B.27]$$
$$X_9 \leq 1 \quad [B.28]$$

Finally, all the $X_j$’s must be nonnegative:

$$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0, X_5 \geq 0, X_6 \geq 0, X_7 \geq 0, X_8 \geq 0, X_9 \geq 0 \quad [B.29]$$

Equations B.17–B.29 represent a linear-programming formulation of this capital-rationing problem.

The optimal solution to this problem is shown in Table B.4. Aspen should adopt in their entirety Projects 1, 3, 4, and 9, and fractional parts of two others—97 percent of Project 6 and 4.5 percent of Project 7. There will be at most one fractional project for each budget constraint; that is, two fractional projects in this problem. The total return (net present value) of the optimal solution is $70.27 or $70,270.

Fractional parts arise from the manner in which the linear-programming model was formulated. By allowing the $X_j$’s to vary from 0 to 1, it was implicitly assumed that the projects were divisible; that is, the firm could undertake all or part of a project and receive benefits (cash flows) in the same proportion as the amounts invested. This assumption is somewhat unrealistic because most investments must either be undertaken in their entirety or not at all. One possible way to eliminate these fractional projects is

<table>
<thead>
<tr>
<th>Primal variables</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
<th>$X_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1^*$</td>
<td>1.0</td>
<td>0</td>
<td>1.0</td>
<td>1.0</td>
<td>0</td>
<td>.970</td>
<td>.045</td>
<td>0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dual variables</th>
<th>$W_1^*$</th>
<th>$W_2^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_1$</td>
<td>.136</td>
<td>1.864</td>
</tr>
</tbody>
</table>

Total net present value ($R^*$) = $70.27 (000)$
to adjust the budget constraints upward to be able to include the entire project. Generally, total capital expenditure limits are flexible enough to allow slight upward adjustments to be made. Another method for eliminating fractional projects in the solution is to use an integer-programming formation of the problem. This would be done by adding constraints to the model that require the $X_j$'s to have integer values:

$$X_j \text{ an integer } j = 1, \ldots, 9$$

that is $X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9$ are integers. Requiring the $X_j$'s to be integers and also to be between 0 and 1 forces these variables to take on the values of either 1 or 0; that is, the projects would have to be accepted either in their entirety or not at all.

The solution to this primal linear-programming problem also yields a solution to the dual problem. There is one dual variable for every constraint in the primal problem. The optimal values of the dual variables associated with the two budget constraints (Equations B.18 and B.19) are shown in Table B.4. In this problem these dual variables indicate the amount that the total present value could be increased if the budget limits (constraints) were increased to permit an additional $1 investment in the given period. In the example, if the budget constraint in Year 1 were increased from $50,000 to $51,000, then the total net present value would increase by $W_1^* = \$136 \times 1,000$ or $\$136$. Similarly, if the budget constraint in Year 2 were increased from $20,000 to $21,000, the total net present value would increase by $W_2^* = \$1.864 \times 1,000$ or $\$1,864$. Because the dual variables measure the opportunity cost of not having additional funds available for investment in a given period, they can be used in deciding whether or not to shift funds from one period to another.\textsuperscript{15} If the values of the dual variables are fairly large, indicating that total net present value could be increased significantly through additional investment, the firm may decide to increase its capital expenditure budget through such methods as new borrowing or equity financing.

**The Transportation Problem: Mercury Candy Company**

Large multiplant firms often produce their products at several different factories and then ship the products to various regional warehouses located throughout their marketing area. The objective is to minimize shipping costs subject to the constraints of meeting the demand for the product in each region and not exceeding the supply of the product available at each plant.

Suppose that the Mercury Candy Company has two production plants located in New England (1) and the Gulf Coast (2), and three warehouses located in the East Coast (1), Midwest (2), and West Coast (3) regions (see Figure B.9). Shipping costs per unit of the product from each of the two plants to each of the three warehouses are shown in the center box in the figure. Demand for the product at each of the regional warehouses is shown in the bottom row and the supply of the product available at each plant is shown in the far right column. The firm wants to minimize its shipping costs.

Begin the linear-programming formulation of the problem by defining $X_{ij}$ to be the number of units of the product shipped from Plant $i$ to Warehouse $j$. This problem has

\textsuperscript{15}The formulation of the capital-rationing problem in a linear-programming framework creates a difficulty in interpreting the dual variables associated with budget constraints. A problem arises because two interdependent measures of the opportunity cost of investment funds are available—the dual variable value and the cost of capital (discount rate), which is used in finding the net present values ($b_j$'s) of the investment projects. For a further discussion of the problem, see William J. Baumol and Richard E. Quandt, “Investment and Discount Rates Under Capital Rationing—A Programming Approach,” *Economic Journal* 75 (June 1965), pp. 317–329.
six $X$-variables—namely, $X_{11}$, $X_{12}$, $X_{13}$, $X_{21}$, $X_{22}$, $X_{23}$. For example, $X_{21}$ indicates the amount of the product shipped from the Gulf Coast plant to the East Coast warehouse. Similar interpretations apply to the other $X$-variables.

Total shipping costs are the sum of the number of units of the product shipped from each plant to each warehouse times the respective shipping cost per unit. The objective function is therefore

$$\text{Min } C = 20X_{11} + 35X_{12} + 65X_{13} + 25X_{21} + 15X_{22} + 50X_{23} \quad \text{[B.30]}$$

There are two sets of constraints (plus nonnegativity constraints) in a standard transportation problem such as this one. The first set has to do with meeting the demand for the product at each of the three regional warehouses. Total shipments to each warehouse must be greater than or equal to demand in the region:

$$X_{11} + X_{21} \geq 2,000 \quad \text{[B.31]}$$
$$X_{12} + X_{22} \geq 1,500 \quad \text{[B.32]}$$
$$X_{13} + X_{23} \geq 1,000 \quad \text{[B.33]}$$

The second set of constraints is concerned with not exceeding the supply of the product at each plant. Total shipments from each plant must be less than or equal to the supply of the product at the plant:

$$X_{11} + X_{12} + X_{13} \leq 2,000 \quad \text{[B.34]}$$
$$X_{21} + X_{22} + X_{23} \leq 2,500 \quad \text{[B.35]}$$

Finally, all the $X$-variables are required to be nonnegative:

$$X_{11} \geq 0, X_{12} \geq 0, X_{13} \geq 0, X_{21} \geq 0, X_{22} \geq 0, X_{23} \geq 0 \quad \text{[B.36]}$$

Equations B.30 through B.36 constitute a linear-programming formulation of the transportation problem.

The optimal solution to this problem is shown in Table B.5. From the New England plant, Mercury should ship 2,000 units to the East Coast regional warehouse ($X_{11}^*$). From the Gulf Coast plant, the firm should ship 1,500 units to the Midwest regional warehouse ($X_{22}^*$) and 1,000 units to the West Coast regional warehouse ($X_{23}^*$). Total shipping costs of the optimal solution are $112,500.

---

16 Special-purpose computational algorithms are available for solving the transportation problem. See Dantzig, *Linear Programming and Extensions*, pp. 308–310, for a discussion of these algorithms.
**Summary**

- Linear-programming problems constitute an important class of constrained optimization problems for which efficient solution techniques have been developed.
- Linear programming has an advantage over classical optimization techniques because it can be applied to problems with inequality constraints.
- Despite the need for expressing the objective and constraint functions as linear relationships, a wide variety of problems can be formulated and solved in the linear-programming framework.
- Virtually all practical linear-programming problems are solved using computer programs that employ algebraic techniques. Graphical solution techniques are used in problems involving two decision variables to illustrate the basic linear-programming concepts.
- An important part of the solution of a linear-programming problem is the value of the dual variables. The dual variables are useful in making marginal resource-allocation decisions. They provide information on the resources that limit the value of the objective function and help make return-versus-cost comparisons in deciding whether to acquire additional resources.

**Exercises**

1. Maytag, a manufacturer of gas dryers, produces two models—a standard (STD) model and a deluxe (DEL) model. Production consists of two major phases. In the first phase, stamping and painting (S & P), sheet metal is formed (stamped) into the appropriate components and painted. In the second phase, assembly and testing (A & T), the sheet metal components along with the motor and controls are assembled and tested. (Ignore any scheduling problems that might arise from the sequential nature of the operations.) Information concerning the resource requirements and availability is shown in the following table:

<table>
<thead>
<tr>
<th>Resource</th>
<th>Quantity of Resources Required per Unit of Output</th>
<th>Dryer Type</th>
<th>Quality of Resources Available During Period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>STD (1)</td>
<td>DEL (2)</td>
</tr>
<tr>
<td>S &amp; P (hours)</td>
<td>1.0</td>
<td>2.0</td>
<td>2,000</td>
</tr>
<tr>
<td>Motors (units)</td>
<td>1</td>
<td>1</td>
<td>1,400</td>
</tr>
<tr>
<td>STD controls (units)</td>
<td>1</td>
<td>0</td>
<td>1,000</td>
</tr>
<tr>
<td>DEL controls (units)</td>
<td>0</td>
<td>1</td>
<td>800</td>
</tr>
<tr>
<td>A &amp; T (hours)</td>
<td>0.333</td>
<td>1.0</td>
<td>900</td>
</tr>
<tr>
<td>Profit contribution ($/unit)</td>
<td>100</td>
<td>125</td>
<td></td>
</tr>
</tbody>
</table>

Each dryer (STD or DEL) requires one motor and a respective control unit. Define $X_1$ and $X_2$ to be the number of STD and DEL dryers manufactured per period, respectively. The objective is to determine the number of STD and DEL dryers to produce so as to maximize the total contribution.

a. Formulate the problem in the linear-programming framework.
2. Suppose that a computer solution of Exercise 1 yielded the following optimal values of the dual variables: $W_1^* = $25 (S & P constraint), $W_2^* = $75 (Motors constraint), $W_3^* = $0 (STD controls constraint), $W_4^* = $0 (DEL controls constraint), and $W_5^* = $0 (A & T constraint). Give an economic interpretation of each of the dual variables.

3. Rework Exercise 1 (a and b), assuming that the profit contributions are $75 for each standard (STD) dryer and $150 for each deluxe (DEL) dryer.

4. The MTA, an urban transit authority, is considering the purchase of additional buses to expand its service. Two different models are being considered. A small model would cost $100,000, carry 45 passengers, and operate at an average speed of 25 miles per hour over the existing bus routes. A larger model would cost $150,000, carry 55 passengers, and operate at an average speed of 30 miles per hour. The transit authority has $3,000,000 in its capital budget for purchasing new buses during the forthcoming year. However, the authority is also restricted in its expansion program by limitations imposed on its operating budget. Specifically, a hiring freeze is in effect and only 25 drivers are available for the foreseeable future to operate any new buses that are purchased. To plan for increased future demand, the transit authority wants at least one-half of all new buses purchased to be the larger model. Furthermore, certain bus routes require the use of the small model (because of narrow streets, traffic congestion, and so on), and there is an immediate need to replace at least five old buses with the new small model. The transit authority wishes to determine how many buses of each model to buy to maximize additional capacity measured in passenger-miles-per-hour while satisfying these constraints. Using the linear-programming framework, let $X_1$ be the number of small buses purchased and $X_2$ the number of large buses purchased.

   a. Formulate the objective function.
   b. Formulate the constraint relationships.
   c. Using graphical methods, determine the optimal combination of buses to purchase.
   d. Formulate (but do not solve) the dual problem, and give an interpretation of the dual variables.


   a. Graph the feasible solution space.
   b. Graph the objective function as a series of isocost lines.
   c. Using graphical methods, determine the optimal solution. Compare the graphical solution to the computer solution given in Figure B.8.

6. Assume that the Agrex Company, a fertilizer manufacturer, wishes to determine the profit-maximizing level of output of two products, Alphagrow ($X_1$) and Better Grow ($X_2$). Each pound of $X_1$ produced and sold contributes $2 to overhead and profit, whereas $X_2$’s contribution is $3 per pound. Additional information:

   • The total productive capacity of the firm is 2,000 pounds of fertilizer per week.
   • This capacity may be used to produce all $X_1$, all $X_2$, or some linear proportion-al mix of the two.
• Because of the light weight and bulk of \( X_2 \) relative to \( X_1 \), the packaging department can handle a maximum of 2,400 pounds of \( X_1 \), 1,200 pounds of \( X_2 \), or some linear proportional mix of the two each week.
• Large amounts of propane are required in the production process. Because of an energy shortage, the firm is limited to producing 2,100 pounds of \( X_2 \), 1,400 pounds of \( X_1 \), or some linear proportional mix of the two each week.
• On the average, the firm expects to have $5,000 in cash available to meet operating expenses each week. Each pound of \( X_1 \) produced requires an initial cash outflow of $2, whereas each pound of \( X_2 \) requires an outflow of $4.

a. Formulate this as a profit-maximization problem in the linear-programming framework. Be sure to clearly specify all constraints.
b. Solve for the approximate profit-maximizing levels of output of \( X_1 \) and \( X_2 \), using the graphical method.

7. Suppose a nutritionist for a United Nations food distribution agency is concerned with developing a minimum-cost-per-day balanced diet from two basic foods—cereal and dried milk—that meets or exceeds certain nutritional requirements. The information concerning the two foods and the requirements are summarized in the following table.

<table>
<thead>
<tr>
<th>Nutrient</th>
<th>Fortified Cereal (Units of Nutrient Per Ounce)</th>
<th>Fortified Dried Milk (Units of Nutrient Per Ounce)</th>
<th>Minimum Requirements (Units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Protein</td>
<td>2</td>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>Calories</td>
<td>100</td>
<td>40</td>
<td>500</td>
</tr>
<tr>
<td>Vitamin D</td>
<td>10</td>
<td>15</td>
<td>400</td>
</tr>
<tr>
<td>Iron</td>
<td>1</td>
<td>0.5</td>
<td>20</td>
</tr>
<tr>
<td>Cost (cents per ounce)</td>
<td>3.0</td>
<td>2.0</td>
<td></td>
</tr>
</tbody>
</table>

Define \( X_1 \) as the number of ounces of cereal and \( X_2 \) as the number of ounces of dried milk to be included in the diet.

a. Determine the objective function.
b. Determine the constraint relationships.
c. Using graphical methods, determine the optimal quantities of cereal and dried milk to include in the diet.
d. Determine the amount of the four nutrients used in producing the optimal diet (\( X_1^* \) and \( X_2^* \)).
e. Based on your answer to part (d), determine the values of the four surplus variables.

8. Suppose that a computer solution of Exercise 7 yielded the following optimal values of the dual variables: \( W_1^* = 0 \) (protein constraint), \( W_2^* = 0 \) (calories constraint), \( W_3^* = 0.05 \) cents (vitamin D constraint), and \( W_4^* = 2.5 \) cents (iron constraint). Give an economic interpretation of each of the dual variables.

9. The government of Indula, in an effort to expand and develop the economy of the country, has been allocating a large portion of each year’s tax revenues to capital investment projects. The Ministry of Finance has asked the various government agencies to draw up a list of possible investment projects to be undertaken over the next three years. The following list of proposals was submitted.
The Ministry of Finance estimates that the present values of the amount (in millions of dollars) available for investment in each of the next three years will be 300, 325, and 350, respectively, and wishes to maximize the sum of the net present value of the projects undertaken.

**a.** Formulate (but do not solve) this problem in the linear-programming framework.

Suppose that a computer solution of this problem yields the following optimal values:

\[
\begin{align*}
X_1^* & = 0.0 & X_2^* & = 0.667 & X_3^* & = 1.0 & X_4^* & = 0.0 & X_5^* & = 0.0 & X_6^* & = 1.0 \\
X_7^* & = 1.0 & X_8^* & = 1.0 & W_1^* & = 0.0 & W_2^* & = 0.667 & W_3^* & = 0.0
\end{align*}
\]

*Note: \(X_1-\ldots-X_8\) are the primal variables (i.e., proportion of each project undertaken) and \(W_1-\ldots-W_3\) are the dual variables associated with the capital budget constraints in each of the next three years.*

**b.** Which of the investment projects should the government undertake? How should we deal with the fractional projects in the optimal solution?

**c.** Give an economic interpretation of the dual variables.

**10.** American Steel Company has three coal mines located in Pennsylvania, Tennessee, and Wyoming. These mines supply coal to its four steel-making facilities located in Ohio, Alabama, Illinois, and California. Monthly capacities of the three coal mines are 6,000, 8,000, and 12,000 tons, respectively. Monthly demand for coal at the four production facilities is 6,000, 5,000, 7,000, and 8,000 tons, respectively. Shipping costs from each of the three mines to each of the four production facilities are shown in the table below:
The company desires to minimize its shipping costs. Let $X_{ij}$ be the amount of coal shipped from Mine $i$ to Production Facility $j$ (for all $i$ and $j$). Formulate (but do not solve) this problem in the linear-programming framework.

11. Mountain States Oil Company refines crude oil into gasoline, jet fuel, and heating oil. The company can process a maximum of 10,000 barrels per day at its refinery. The refining process is such that a maximum of 7,000 barrels of gasoline can be produced per day. Furthermore, there is always at least as much jet fuel as gasoline produced. The wholesale prices of gasoline, jet fuel, and heating oil are $50, $40, and $30 per barrel, respectively. The company is interested in maximizing revenue.

   a. Formulate (but do not solve) this problem in the linear-programming framework. **Hint:** There should be three constraints in this problem.

   b. Suppose that a computer solution of this problem yields an optimal value of $45 for the dual variable associated with the capacity constraint of the refinery. Give an economic interpretation of this dual variable.

12. Given the following profit-maximization problem:

   \[
   \text{Max } 10X_1 + 9X_2 \\
   \text{subject to: } \begin{align*}
   &.70X_1 + X_2 \leq 630 \quad \text{Raw materials constraint (units)} \\
   &.50X_1 + .833X_2 \leq 600 \quad \text{Skilled labor constraint (hours)} \\
   &X_1 + .667X_2 \leq 708 \quad \text{Unskilled labor constraint (hours)} \\
   &.10X_1 + .25X_2 \leq 135 \quad \text{Storage capacity constraint (units)} \\
   &X_1, X_2 \geq 0
   \end{align*}
   \]

   where $X_i =$ amount of product $i$ manufactured per period, and the following optimal computer solution:

   \[
   \begin{array}{c|c|c|c|c|c}
   \text{Variable} & \text{Value} & \text{Variable} & \text{Value} \\
   \hline
   X(2) & 252 & W(1) & 4.375 \\
   X(4) & 120 & W(2) & 0 \\
   X(1) & 540 & W(3) & 6.9375 \\
   X(6) & 18 & W(4) & 0 \\
   \end{array}
   \]

   determine the following:

   a. Amount of each product that should be manufactured each period
   b. Maximum total profit
   c. Amount of each resource used to obtain the optimal output
d. Amount that total profit will change if one additional unit of raw material is made available to the manufacturing process

e. Amount that total profit will change if skilled labor input is increased by one hour

13. How could your university make use of linear programming to schedule courses, professors, and facilities? Try to develop explicitly the objective function and to identify the important constraints within such a linear-programming model.

14. Obtain a copy of a management science textbook (such as the Anderson et al. book referenced in footnote 1 of this chapter) that describes and illustrates the step-by-step procedure involved in solving linear-programming problems using the simplex method. After studying this algorithm, apply it to one or more of the linear-programming problems presented above (such as 1, 4, 6, and 7). Compare your answers with the graphical solutions.

Note: The following exercises require the use of a computer program to solve the linear-programming problems.

15. Solve Exercise 1 using a computer program. Compare the computer solution with the graphical solution.

16. Solve Exercise 4 using a computer program. Compare the computer solution with the graphical solution.

17. Solve Exercise 6 using a computer program. Compare the computer solution with the graphical solution.

18. Solve Exercise 7 using a computer program. Compare the computer solution with the graphical solution.

19. Solve Exercise 9 using a computer program.

20. Solve Exercise 10 using a computer program.

21. Solve Exercise 11 using a computer program.

22. Access the LINDO Systems, Inc. Internet site at http://www.lindo.com/. There are several different programs that you can download; the activity below is based on downloading the What's Best! software, which plugs in to Excel. Once you download What's Best!, double-click on the downloaded file (the one used here is wb30) to install the program. Once the program is installed, you can learn how to use What's Best! by accessing “A Simple Tutorial” by way of the “What's Best! Help” file. What's Best! comes with a number of sample linear programming sample spreadsheets, including a cost-minimizing shipping problem and a profit-maximizing product mix problem.