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A Unified Approach to Incorporating Demographic or Other Effects into Demand Systems

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A general method of introducing demographic effects into any demand system, using modifying functions, is described which permits complicated interactions of demographic variables with prices and expenditures. Theorems give properties the modifying functions must have to ensure integrability of the resulting system. Demand equations of the new system are given explicitly as functions of the original demand equations and the modifying functions. The procedure is interpreted as altering a household's technology, and is shown to encompass adult equivalent scales and related methods. Examples of modifying functions are derived and applications of the technique to uses other than demographic variation are considered.

INTRODUCTION

The oldest and most commonly used method of introducing demographic variation into demand equations is with adult equivalent scales. The problem with equivalent scales, including the more general commodity specific scales, is that they only permit a very restricted range of demographic effects. Specifically, changes in demographic characteristics are forced to be virtually equivalent to changes in prices. This rules out complicated interactions of demographic effects with prices or total expenditures, as well as eliminating many potential models that are attractive in their simplicity. The few extensions to equivalent scales that have been proposed, such as demographic overhead or translation terms, help to alleviate this problem, but are still quite restrictive forms.

The main alternative to equivalent scales is to take a specific demand equation or system, and let some of its parameters vary demographically. This is the approach taken by Stoker (1979). This is also done implicitly by researchers who include demographic variables on the right hand side of ad hoc models of the demand for individual goods. This procedure allows for virtually any set of interactive demographic and price effects, but does not have any general applicability, being specific to the given starting model.

This paper describes a large middle ground between the extremes of individual model modification and equivalent scales. What is given is a large class of possible modifications have both the universal applicability of demographic scaling and translating, and the flexibility to allow demographic variables to interact with price and expenditure terms in the demand system in an almost unlimited variety of ways.

The method is to introduce functions of demographic variables, prices, and expenditures into the cost (expenditure) function of a demand system. Section 1 gives properties these functions must have to guarantee the integrability of the modified system, regardless of what initial system the demographically varying modifying functions were applied to. Also shown is how the modified demand system may be written directly as a function of the original system, so the effect of modifying functions on the demand equations may be directly assessed, without consideration of the cost or utility functions involved.
In Section 2, it is shown how the modifying functions may be interpreted as representing household technologies. The modified utility function equals the original utility function evaluated at the values of demographically varying intermediate goods. In this light, it is seen that the modification technique presented here is a natural extension of models described by Gorman (1976). Gorman generalized commodity specific equivalent scales into linear household technologies, and also gave a few specific examples of nonlinear technologies. Modifying functions extend this generalization, by describing how demographically varying, nonlinear technologies may be introduced into demand systems. The intermediate good production implied in homothetically separable (two-stage maximization) utility functions is also encompassed by modifying functions. The demonstration that equivalent scales, demographic translation, Gorman's model, and homothetically separable utility are all special cases of modifying functions is given in Section 3. The relationship of modifying functions to models that specify classes of cost functions, like Muellbauer's (1975) generalized linearity, is also discussed in Section 3.

The remaining sections are devoted to applications of modifying functions. In Section 4, it is shown how one may demographically scale and translate budget shares of goods, instead of quantities. This is a natural way of introducing demographic variation into models like the AIDS or the indirect translog system, which are fundamentally budget share models. Also considered are models where the translation or scaling terms are functions of prices and expenditures as well as demographic variables. As an example, one such “generalized translation” model has demographically varying committed (or overhead) quantities that are price sensitive, rather than constant.

This last example demonstrates an application of the technique to purposes other than the introduction of demographic or other variables. Modifying functions may be used to endow demand systems with new properties. For example, just as ordinary translation introduces constant committed quantities into demand systems (turning a Cobb-Douglas into a linear expenditure system, for example), the generalized translation model given at the end of Section 4 introduces price sensitive committed quantities into demand systems.

In Section 5, we consider generalized demographic scaling and translating of quantities. Expenditures or prices cannot be put directly into ordinary equivalent scales or translation terms, because doing so would put expenditures or prices into the utility function itself. However, using modifying functions, any demand system can be modified to contain scaling and translation terms that are functions of prices and expenditure level as well as demographic or other variables. Section 5 includes examples of these types of modifying functions.

Section 6 is a summary, and includes suggestions for further work and other applications.

1. THE TECHNIQUE OF COST FUNCTION MODIFICATION

Let $U^*(q^*)$ be a direct utility function of an $n^*$ vector of quantities $q^*$. $U^*$ is assumed to be continuous, non-decreasing, and quasiconcave in $q^*$. Let $V^*(m^*, p^*)$ be the corresponding indirect utility function of total expenditures $m^*$ and $n^*$ vector of prices $p^*$. $V^*$ may be inverted to give the cost function $C^*(u, p^*)$ for utility level $u$. Let Hicksian demands from $C^*$ be given by $H^*_*(u, p^*)$, Marshallian demands by $D^*_*(m^*, p^*)$, and Marshallian demands in budget share form, $p_i^*D^*_i/m^*$, by $W^*_i(m^*, p^*)$. 


Define a function $C(u, p)$ to be a legitimate cost function if the Marshallian demands derived from it come from a utility function with the above listed properties. Specifically, any function $C(u, p)$ that is homogeneous of degree one in prices, nonnegative, non-decreasing in prices, increasing in $u$, increasing in at least one price, and is concave in prices is a legitimate cost function. As a result of these properties, cost functions are also continuous in prices $p$, and have first and second derivatives with respect to $p$ that exist everywhere except possibly on a set of measure zero.

Let $C^*$ be a legitimate cost function, $p$ be an $n$ vector of prices, and $r$ be any vector of demographic variables. Consider a new cost function $C$, given by the general transformation

$$C(u, p, r) = f[C^*(u, h(p, r)), p, r].$$

Let $m^* = C^*[u, h(p, r)]$ and $p_i^* = h_i(p, r)$. Vectors of values or functions are denoted here by dropping subscripts, so $p^* = h(p, r)$. It is assumed throughout this paper that $h_i \geq 0$ for all $i$ and strictly positive for at least one $i$, and that $f$ and $h$ are continuous and have first and second derivatives that exist everywhere, except possibly on a set of measure zero. The $f$ function in equation (1) permits interaction of demographic variables with expenditures, while all the $f$ and $h$ functions allow interactions of $r$ with $p$.

A set of restrictions on the $f$ and $h$ functions will be derived which guarantee that $C$ will be a legitimate cost function. These restrictions are maximal, or necessary, in the sense that if any of them are violated, there will exist a legitimate $C^*$ for which $C$ will not be legitimate. The Marshallian demand system $D$ arising from $C$ will be given as an explicit function of the original demand system $D^*$, thereby facilitating application of the technique in practice, when $C^*$ itself may not be tractible, or even known. What follows are three theorems describing the required set of restrictions on $f$ and $h$. Proofs of theorems are in the Appendix.

**Theorem 1.** Let $C^*$ be a legitimate cost function, and let $C$ be as given in equation (1). Then $C$ is homogenous of degree one in prices if $f$ and $h$ have the properties that for some function $\theta(p, r)$,

$$\sum_{j=1}^{n} \frac{\partial h_i(p, r)}{\partial p_j} \frac{p_j}{p_i^*} = \theta(p, r), \quad (i = 1, \ldots, n^*)$$

$$1 - \sum_{j=1}^{n} \frac{\partial f(m^*, p, r)}{\partial p_j} \frac{p_j}{f(m^*, p, r)} = \frac{\partial f(m^*, p, r)}{\partial m^*} \frac{m^*}{f(m^*, p, r)} \theta(p, r).$$

Furthermore, if $f$ and $h$ do not satisfy equations (2) and (3) for some function $\theta(p, r)$, then there exists at least one legitimate cost function $C^*$ such that the corresponding function $C$ is not homogenous of degree one in prices.

**Theorem 2.** Let $C^*$ be a legitimate cost function, and let $C$ be as given in equation (1). Then $C$ is nonnegative, nondecreasing in prices, increasing in $u$, and increasing in at least one price if $f(m^*, p, r) \geq 0$, $\frac{\partial f(m^*, p, r)}{\partial m^*} > 0$, for all $i$ $\frac{\partial f(m^*, p, r)}{\partial p_i} \geq 0$, for all $i$ and $j$ $\frac{\partial h_i(p, r)}{\partial p_j} \geq 0$, and either there exists an $i$ such that $\frac{\partial f(m^*, p, r)}{\partial p_i} > 0$ or there exists a function $\tilde{f}(j)$ such that, for all $j$, $\frac{\partial h_j(p, r)}{\partial p_{i(j)}} > 0$. Furthermore, if $f$ or $h$ violate any of these conditions then there exists at least one cost function $C^*$ such that the corresponding function $C$ will not possess all of the above properties.
Theorem 3. Let $C^\ast$ be a legitimate cost function and let $C$ be as given in equation (1). Assume $\partial f/\partial m^\ast \geq 0$ and $h \geq 0$. Let $B_1$, $B_2$, and $B_3$ be defined by:

$$B_1(i, m^\ast, p, r, v) = \frac{\partial^2 f(m^\ast, p, r)}{\partial m^\ast^2} \left( \frac{\partial h(p, r)}{\partial p} \right)_v, \quad (i = 1, \ldots, n^\ast)$$

$$B_2(i, m^\ast, p, r, v) = \frac{\partial f(m^\ast, p, r)}{\partial m^\ast} \left[ v \frac{\partial^2 h(p, r)}{\partial p \partial p'} \right]_v, \quad (i = 1, \ldots, n^\ast)$$

$$B_3(m^\ast, p, r, v) = \frac{1}{n^\ast} \left[ v \frac{\partial^2 f(m^\ast, p, r)}{\partial p \partial p'} \right]_v$$

(4) (5) (6)

for arbitrary $n$ vector $v$. Then $C$ is concave in $p$ if $B_1 \leq 0$, $B_2 \leq 0$, and $B_3 \leq 2\sqrt{B_1 B_3}$ for all $i = 1, \ldots, n^\ast$, $m^\ast \geq 0$, $P \geq 0$ and $v$. Furthermore, if $f$ and $h$ are such that $B_1$, $B_2$, and $B_3$ do not have these properties, then there exists at least one legitimate cost function $C^\ast$ for which the corresponding function $C$ is not concave in $p$.

The interpretation of Theorem 3 comes from observing that $B_1 \leq 0$ if and only if $f$ is concave in $m^\ast$, $B_2 \leq 0$ for a given $i$ if and only if $h_i$ is concave in $p$, and $B_3 \leq 0$ if and only if $f$ is concave in $p$. So, Theorem 3 says that $f$ must be separately concave in $m^\ast$ and $p$, while $h$ cannot be too convex in $p$ relative to the concavity of $f$.

Theorems 1, 2, and 3 together show the properties a set of functions $f$ and $h$ must have to guarantee that for any legitimate cost function $C^\ast$, the function $C$ given by equation (1) will also be legitimate. Define a set of functions $f$ and $h$ to be Modifying Functions if $f$ and $h$ satisfy Theorems 1, 2, and 3, have $h_i \geq 0$ for $i = 1, \ldots, n^\ast$, and have existing first and second derivatives. Correspondingly, given a starting cost function $C^\ast$, the function $C$ given by equation (1) is defined as a modified cost function, the set of demands arising from $C$ is a modified demand system, and these demands come from a modified direct and modified indirect utility function.

It should be stressed that, for any particular $C^\ast$, a legitimate $C$ could result from the application of an $f$ and $h$ that are not modifying functions. The properties given by Theorems 1, 2, and 3 are necessary only in the sense that, for each property, there exists a legitimate $C^\ast$ that will not yield a legitimate $C$ if $f$ or $h$ violate that property. Therefore, no weaker set of restrictions are possible if this technique is to be as universally applicable as demographic scaling or translating. The sufficiency of these properties permits use of the technique to modify any legitimate demand system. For any legitimate $C^\ast$, any $f$ and $h$ that are modifying functions will yield a legitimate $C$.

The next step is to describe the modified demand system. First, define the function $F$ as the inverse of $f$ on its first element, that is, $F(m, p, r) = m^\ast$. By Theorem 2, $f$ is monotonically increasing in its first element, so the function $F$ is well defined. For the rest of this paper, $m^\ast$ and $p^\ast$ can be replaced by $F(m, p, r)$ and $h(p, r)$, respectively, so all equations are in terms of observable variables $m, p,$ and $r$. The following theorem describes the modified indirect utility function and demand system in terms of the starting system’s indirect utility function and demands. The indirect utility function is simply derived from definitions and the fact that $C = M$ and $C^\ast = M^\ast$. Demand curves are then obtained in the usual way.

Theorem 4. Let $C^\ast$ be a legitimate cost function, $f$ and $h$ be modifying functions, and $C$ be a modified cost function. Then the modified indirect utility function is given by

$$V(m, p, r) = V^\ast(m^\ast, p^\ast).$$

(7)
The modified Hicksian demands are given by

\[ H_i(u, p, r) = \frac{\partial f(C^*(u, p^*), p, r)}{\partial C^*(u, p^*)} \sum_{j=1}^{n^*} \frac{\partial h_j(p, r)}{\partial p_i} H_j^*(u, p^*) + \frac{\partial f(C^*(u, p^*), p, r)}{\partial p_i}, \quad (i = 1, \ldots, n), \] (8)

The modified Marshallian demands are given by

\[ D_i(m, p, r) = \frac{\partial f(m^*, p, r)}{\partial m^*} \sum_{j=1}^{n^*} \frac{\partial h_j(p, r)}{\partial p_i} D_j^*(m^*, p^*) + \frac{\partial f(m^*, p, r)}{\partial p_i}, \quad (i = 1, \ldots, n) \] (9)

and, in budget share form, modified Marshallian demands are

\[ W_i(m, p, r) = \frac{\partial f(m^*, p, r)}{\partial m^*} \frac{m^*}{m} \sum_{j=1}^{n^*} \frac{\partial h_j(p, r)}{\partial p_i} \frac{p_i}{p_j} W_j^*(m^*, p^*) + \frac{\partial f(m^*, p, r)}{\partial p_i} \frac{p_i}{m^*}, \quad (i = 1, \ldots, n). \] (10)

Theorem 4 shows how modified demand systems can be made, and their integrability be assured, without actually constructing the cost or utility function of either the original or the modified system.

The next theorem demonstrates the generality of modifying functions, by showing that any nondemographically varying starting cost function can be transformed into any demographically or nondemographically varying cost function by the appropriate choice of modifying functions. This also shows that equation (1) imposes no restriction on cost functions even when \( f \) and \( h \) are restricted to be modifying functions.

**Theorem 5.** Let \( C^*(u, p) \) be any legitimate cost function that varies only with \( p \) and \( u \), and let \( C(u, p, r) \) be any legitimate cost function that varies only with \( p, r \), and \( u \). Then there exists modifying functions \( f \) and \( h \) such that the cost functions \( C(u, p, r) \) and \( f[C^*(u, h[p, r]), p, r] \) represent the same set of preferences.

Before proceeding to interpretation and examples of modifying functions, brief consideration should be given to the issues of estimation of a modified system. First, there is no guarantee that all parameters in a transformed demand system will be identified, just as there is no guarantee that all the parameters in any demand system will be identified. For example, if \( D_i^* \) has a constant term and \( f \) is linear in \( m \) and \( p \), then \( D_i \) will have two constant terms, only the sum of which may be identified.

For hypothesis testing, the cost function modification technique may be useful for testing the adequacy of models. Consider functional form specifications of \( f \) and \( h \) that have \( f = m \) and \( h = p \) nested within them as special cases. For all such forms of \( f \) and \( h \), the modified system \( D \) has the original system \( D^* \) nested within it. Likelihood ratio tests of \( D^* \) versus various \( D \) systems provide checks of the adequacy of \( D^* \) against the more general \( D \) models.

Another point is that the \( r \) parameters need not be directly observed demographic variables. The vector \( r \) can be a function of demographic variables, with parameters that
are estimated along with those of the demand system, such as Pollak and Wales (1981) linear scaling and translation functions. The vector \( r \) could also represent random variation across consumers, or could be observations of stocks of durables.

When \( r \) is a vector of fixed parameters that does not vary across consumers, modifying functions provide a convenient way of constructing new demand systems as variants of known systems. Unlike directly specifying variations of known utility functions, the use of modifying functions allows new demand equations to be explicitly described as functions of the starting demand system equations.

2. INTERPRETING THE MODIFYING FUNCTIONS

Modifying functions may be interpreted as indirect specifications of the correspondence \( q^* = g(q, r) \). Theorem 6 below shows how \( g \) is formed. With this construction the variables \( q^* \) are thought of as quantities of intermediate goods, such as prepared meals or transportation, from which a household derives utility. The equation \( q^* = g(q, r) \) then shows how the intermediate goods are constructed from input goods \( q \). All households are interpreted as having the same utility function \( U^* \) with respect to intermediate goods \( q^* \). Differences across households are embodied in \( r \) and hence in \( g \), which are demographically varying household technologies. This \( g \) describes how each type of household produces intermediate goods from purchased input goods. The relationship of \( h \) to \( f \) and \( g \) is that \( h \) is proportional to the shadow prices of the intermediate goods in \( U^* \), while \( f \) specifies an overhead or necessities technology within \( g \).

**Theorem 6.** Let \( q \) be the quantities demanded by a consumer with a modified demand system \( D(m, p, r) \) faced with prices \( p \) and total expenditure level \( m \), and let \( q^* \) be the quantities demanded by a consumer with the starting demand system \( D^*(m^*, p^*) \) faced with prices \( p^* = h(p, r) \) and total expenditure level \( m^* = f(m, p, r) \). Let \( \phi(q, r) = p/m \) be the inverse demand correspondence of the modified demand system \( D(m, p, r) \). Then \( q \) and \( q^* \) are related by the consistent system

\[
\begin{align*}
q - \frac{\partial f[F(1, \phi(q, r), r), \phi(q, r), r]}{\partial \phi(q, r)} &\bigg/ \frac{\partial f[F(1, \phi[q, r], r), \phi(q, r), r]}{\partial F(1, \phi[q, r], r)} \\
= \left[ \frac{\partial h[\phi[q, r], r]}{\partial \phi(q, r)} \right] q^*.
\end{align*}
\]

(11)

Also, the correspondence \( q^* = g(q, r) \) given by equation (11) is a function over the set of \( q \) such that \( \phi(q, r) \) is a function and rank \( (\partial h/\partial \phi) \cong n^* \).

Although \( q^* = g(q, r) \), \( U^*(g(q, r)) \) and \( U(q, r) \) are not necessarily identical functions. However, by equation (7) we have the standard duality result that \( \max (U^*(q^*))p^*q^* = m^* \) = \( \max (U(q, r)|pq = m) \). Therefore, evaluated at the \( Q \) that (constrained by \( pq = m \)) maximizes \( U(q, r), U^*(g(q, r)) \) and \( U(q, r) \) are equal in value.

This duality result shows that the \( h_i(p, r) \) functions are proportional to intermediate good shadow prices, since maximizing \( U(q, r) \) given \( p \) and \( m \) yields the same utility as directly buying intermediate goods \( q^* \) that maximize \( U^*(q^*) \) under price regime \( (m^*/m)h(p, r) \) and the same total expenditure level \( m \).

For the remainder of this section, we will consider \( h \) and \( f \) separately, to analyse their separate effects on the demand system. Starting with \( h \), we have that when \( f \) is the identity so \( m^* \) equals \( m \), \( h_i \) are precisely the shadow prices of the intermediate goods.
From equation (11) when \( f \) is the identity we also have that \( g = (\partial h / \partial p)^{-1} q \), for some generalized inverse \(-1\). So, for a given price regime \( p \), the matrix of \( \partial h_i / \partial p_j \) derivatives is a demographically varying intermediate good technology which is identical to that described by Gorman (see Section 3) when \( h(p, r) \) is linear in \( p \). When \( h(p, r) \) is nonlinear, the function \( g \) relationship embodies classical (intermediate good) production functions, though without overheads.

Now consider the effect of \( f \) alone, by letting \( h = p \) so \( U^*(q^*) \) is maximized given \( p'q^* = m^* \). It is as if \( q^*_i \) has the same price as \( q_i \) but the total expenditures available for buying \( q^*_i \) is \( m^*_i \), rather than the \( m_i \) that is available for buying \( q \). This is consistent with the notion of \( f \) embodying some kind of money depleting overhead technology. For example, if \( q^*_i = q_i + r_i \), then maximizing \( p'q \) given \( m \) is equivalent to maximizing \( q^*_i \) given \( m^*_i = m - p'r_i \). This is demographic translation. In this example, \( r \) corresponds to a vector of committed quantities, which may equivalently be interpreted as overheads in the household production function.

When \( h = p \), the relationship of \( q \) to \( g \) is \( q = (\partial f / \partial p) + (\partial f / \partial m^*) q^* \). The \( f \) transformation is seen to have two components. The \( \partial f / \partial p \) terms correspond to overheads, or the minimum amount of each input that must be obtained before "production" begins. These required overheads may vary with prices and expenditures, in which case we may think of them as describing a separate set of technologies required to augment the basic technologies of the other terms. This is a logical extension of the description of translation terms as committed expenditures required before (or in addition to) utility production. The same idea is embodied in what Blackorby, et al. (1978, p. 284) call a "price dependent subsistence bundle".

The \( \partial f / \partial m^* \) term rescales all intermediate goods uniformly, while \( m^* \) affects both the relative and absolute amounts of the intermediate goods \( q^* \). Intuitively, the larger the overheads given by \( \partial f / \partial p \) are, the smaller the intermediate good production level \( q^* \) must be for a given \( m \). In fact, total supernumerary expenditures exactly equal \( m^*(\partial f / \partial m^*) \). This is shown by rearranging equation (3) (with \( \theta = 1 \) because \( h = p \)) to get \( m^*(\partial f / \partial m^*) = m - p'(\partial f / \partial p) \). Here \( m^* \) and \( \partial f / \partial m^* \) combine to exactly offset the cost of the overhead technologies \( \partial f / \partial p \).

3. MODIFYING FUNCTIONS AND OTHER TECHNIQUES

This section discusses how modifying functions compare to other techniques of introducing demographic variables into demand systems, and to other general functional equation specifications of cost functions. First it will be shown that equivalent scales and related models are all special cases of modifying functions. Engel (1895) style noncommodity specific scaling is given by \( h = p \) and \( f = m^*/r \) where \( r \) is in this case a simple scalar giving the number of equivalent adults in the household. Barten (1964) style commodity specific equivalent scales is given by \( h_i = r_ip_i \) and \( f = m^* \) where \( r_i \) is the number of equivalent adults for good \( i \) in the household.

Modifying a demand system to incorporate commodity specific adult equivalent scales is what Pollak and Wales (1981) call demographic scaling. Their other general procedure, demographic translating, is given by \( h = p \) and \( f = m^* + p'r \) for an \( n \) vector of demographic variables \( r \).

What Pollak and Wales call the Gorman form and reverse Gorman form are also special cases of modifying functions, since they simply combine the above scaling and translating specifications. Gorman (1976) actually describes more general models than these, including a model of linear household technologies. In this model, not only is
there demographic translating and scaling, there are also joint demographically varying effects across commodities. Gorman supports this model by the example of "a selfish father who nevertheless found it socially necessary to buy his wife flowers and his children toys, for instance, each time he bought himself a new golf club". This model is also obtainable with modifying functions, including the extension for when the number of outputs does not equal the number of inputs. General linear household technologies are given by

\[ h_i(p, r) = \sum_j r_{ij} p_j, \quad (i = 1, \ldots, n) \]  

\[ f(m, p, r) = m - \sum_j s_j p_j \]  

where \( r \) now contains the \( r_{ij} \)'s and \( s_j \)'s which all may vary demographically.

The modifying function technique is another step in the string of generalizations just described. It represents linear or nonlinear demographically varying household technologies, as described in Section 2.

An alternative approach to introducing demographic variables via some transformation of a cost function is to select some parameters from a given cost function and let them vary demographically. This approach lacks the general applicability of modifying functions, since it is completely dependent on the exact functional form of the chosen demand system. This approach also lacks any general interpretation such as that of an intermediate utility production function as described in Chapter 2. Even so, given any particular cost function that has been modified by assuming its parameters vary demographically, this same transformation must also be possible using modifying functions by virtue of Theorem 5, although the form of the modifying functions required to do the transformation may be complicated.

Modifying functions may also be compared to techniques that specify cost functions as functional equations. For example Muellbauer (1975) defines Generalized Linearity (GL) as cost functions of the form \( C(u, p) = G_1[u, G_2(p)]G_3(p) \) for some functions \( G_1, G_2, \) and \( G_3 \). Demographic variation may be introduced by allowing \( G_2 \) and \( G_3 \) to vary with \( r \). Price Independent Generalized Linearity (PIGL) restricts \( C \) further by assuming the form \( C(u, p) = [H_1(p)^e + uH_2(p)^e]^{1/e} \). Pollak's (1972) generalized separability specifies indirect utility functions that are of the form \( \sum_i G_i[\log (p_i/m) - H_i(p_i/m)] - H_3(p/m), \) except for some special cases that are of a slightly different form. Pollak and Wales (1981) describe what they call the generalized Prais–Houthakker (1955) form for incorporating demographic variables into a demand system. This form imposes direct additivity of the utility function.

Although some of these models resemble equation (1), they all differ from modifying functions in one fundamental way. These models all consist of imposing restrictions on preferences by constraining cost or utility functions to take on certain forms. Modifying functions impose no such restrictions. As Theorem 5 shows, all of the above listed models must be describable as modified cost or utility functions.

Rather than specifying a class of cost functions with certain properties, modifying functions are a class of cost function transformations that preserve certain properties. By imposing the restrictions of Theorems 1, 2, and 3, the modified cost function inherits the properties of the starting cost function that make it legitimate. In some cases, the additional structure of one or more of the previously cited cost function specifications may also be inheritable. For example, if \( f(m^*, p, r) = m^* \), then it is straightforward to show that a modified cost function will inherit the property of being GL if the starting cost function is GL.
One type of demand specification that does deal with inheritance of cost or utility function properties is separability involving two stage budgeting. Let \( f(m^*, p, r) = m^* \), and assume that each price \( p_i \) appears in one and only one of the functions \( h_i(p) \) for \( i = 1, \ldots, n \). Then equation (1) has the form

\[
C(u, p) = C^*[u, h_1(P_1), \ldots, h_n*(P_n*)]
\]  

(13)

where \( P_i \) is the vector of prices that appear only in the function \( h_i \). Equation (13) is identical to equation (7) in Blackorby, Primont and Russell (1978, p. 122). They show that a certain form of separability in \( C(u, p) \) implies that \( C(u, p) \) can be written in the form of equation (13), where \( C^* \) inherits the properties of a cost function, and \( h_i(p) \) also inherit certain properties. As a special case of modifying functions, equation (13) shows that inheritance also goes the other way, that is, given the properties of \( C^* \) and \( h_i \) the function \( C \) inherits the properties of a cost function.

Blackorby, Primont and Russell (1975, 1976) have proved that the structure of equation (13) implies that the utility function \( U(q) \) is homothetically separable, that is, \( U(q) = U^*[U_1(Q_1), \ldots, U_n(Q_n)] \) where \( Q_i \) is a vector of quantities corresponding to prices \( P_n \) and \( U_i \) is a homothetic function. This structure is precisely what is required for price indices \( p^* \) and quantity indices \( q^* \) to exist such that \( u(q) \) can be maximized by first maximizing \( U^*(q^*) \) subject to \( m = p^*q^* \), then maximizing \( g_i(Q_i) \) subject to \( p^*q^* = P_iQ_i \) for each \( i \) to obtain \( Q \). Two stage maximization is a special case of the intermediate good production functions of Section 2, where \( U_i(Q_i) = g_i(q) \). The theorem that equation (13) implies homothetically separable utility is a statement linking properties of \( f \) and \( h \) to properties of \( g \) in the special case of modifying functions given by equation (13).

Using equation (13) to incorporate demographics into a demand system, so \( p^*_i = h_i(p_i, r) \), corresponds in the utility function to two stage budgeting where the first stage utility function \( U^*(q^*) \) is the same for all households while the second stage functions \( g_i(Q_i, r) \) differ demographically. This model has the attractive property of assuming that households can be similar in terms of their demands for aggregate commodities like shelter, but differ in terms of details like the choice between buying margarine or butter.

4. AN APPLICATION: TRANSLATING AND SCALING OF BUDGET SHARES

As Pollak and Wales (1981) have summarized, two ways of introducing demographic effects are through demographic scaling, which is just commodity specific adult equivalent scales; and demographic translating, which is basically demographically varying constant terms in the demand equations. Some demand systems have constant terms in the equations for budget shares instead of in the quantity equations. An example is the Almost Ideal Demand System of Deaton and Muellbauer (1980b). If these constants were to vary demographically, we would essentially have demographic budget share translation. An example of this kind of model is the demographic variation in a homothetic indirect translog demand system used by Stoker (1979). Demographic budget share scaling can be defined analogously.

In this section, the use of modifying functions to do budget share scaling and translating is analysed. A theorem giving the exact properties modifying functions \( f \) and \( h \) must have to demographically scale and translate budget shares is given and generalized scaling and translating of budget shares is defined.

The following theorem describes the class of modified budget share systems that, like scaling and translating of quantities, operate separately on each good.
Theorem 7. Let \( n = n^* \), and let a modified demand system in budget share form be given by

\[
W_i(m, p, r) = T_i(W_i^*(m^*, p^*), m, p, r), \quad (i = 1, \ldots, n)
\]

where the function \( T_i \) is independent of \( W_i^* \) for all \( j \neq i \), for \( i = 1, \ldots, n \). Then \( T_i \) has the form

\[
W_i(m, p, r) = S(m, p, r) W_i^*(m^*, p^*) + R_i(m, p, r) \quad (i = 1, \ldots, n)
\]

where, for some modifying functions \( f \) and \( h \),

\[
R_i(m, p, r) = \left[ \frac{\partial f(m^*, p, r)}{\partial p_i} \right] p_i / m,
\]

\[
S(m, p, r) = 1 - \sum_{i=1}^{n} R_i(m, p, r), \quad \text{and} \quad \frac{\partial h_i(p, r)}{\partial p_j} = 0 \quad \text{for all} \quad j \neq i.
\]

By analogy with Pollak and Wales’ terminology, the functions \( R_i \) can be defined as generalized (commodity specific) budget share translation terms. They are generalized in the sense that they vary with \( m \) and \( p \), and not just with demographic variables. Similarly, the function \( S \) is a generalized budget share scale term. Theorem 7 shows that such scale terms cannot be commodity specific, and that they can only exist when translation terms are also present.

Analogous to quantity scaling and translating, define ordinary budget share scaling and translating as budget share transformations of the form

\[
W_i(m, p, r) = r_i + n W_i^*(m^*, p^*) + r_i, \quad (i = 1, \ldots, n).
\]

The following theorem completely characterizes the class of modifying functions that yield budget share transformations of the form of equation (16).

Theorem 8. Modified budget share equations are of the form of equation (16), that is, ordinary budget share scaling and translation, if and only if modifying functions \( f \) and \( h \) are of the form

\[
f(m^*, p, r) = \beta(r) m^* \left[ s(r) / \alpha(r) \right] \tilde{P}(p, r)
\]

\[
h_i(p, r) = \gamma(r) p_i^\alpha(r)
\]

where \( r_i \geq 0 \) for \( i = 1, \ldots, n \), \( s(r) = 1 - \sum_{i=1}^{n} r_i \), \( 0 < s(r) \leq 1 \), \( \tilde{P}(p, r) = \prod_{i=1}^{n} P_i^{r_i} \) and \( \alpha, \beta, \) and \( \gamma \) are functions satisfying \( \gamma(r) > 0 \), \( \beta(r) > 0 \), and \( s(r) \leq \alpha(r) \leq 1 + \delta(r) + \sqrt{\delta(r)^2 + 2 - 2s(r)} ) \delta(r) \) where \( \delta(r) = (2/n) \min_i [r_i / (s + r_i)] \). Also, the modified budget shares have the form

\[
W_i(m, p, r) = s(r) W_i^*(m^*, p^*) + r_i, \quad (i = 1, \ldots, n)
\]

where \( p^* = h(p, r) \) and

\[
m^* = m \left[ \beta(r) \tilde{P}(p, r) \right]^{-\left[ \alpha(r) / s(r) \right]}.
\]

A simple example of ordinary budget share translating and scaling is to let \( \alpha(r) = s(r) = a \) where \( a \) is a positive constant less than one, and \( \gamma(r) = \beta(r) = 1 \). This gives the modified system

\[
W_i(m, p, r) = (1 - a) W_i^*[m / \tilde{P}(p, r)^{(1/a)}] P_i^a, \ldots, P_n^a] + r_i, \quad (i = 1, \ldots, n)
\]

which requires \( r_i \geq 0 \) and \( \sum_{i=1}^{n} r_i = a \).

An example of generalized scaling and translating of budget shares is given by

\[
F(m^*, p, r) = a \tilde{P}(p, r)^{(1/a)} + m^* \quad \text{and} \quad h(p, r) = p, \quad \text{where} \quad \tilde{P}, \quad a, \quad \text{and} \quad r_i \quad \text{are as above. It is straightforward to verify that this} \quad f \quad \text{and} \quad h \quad \text{are modifying functions, and the modified budget share system they correspond to is given by}
\]

\[
W_i(m, p, r) = \left[ m / m^* \right] W_i^*(m^*, p) + r_i \frac{\tilde{P}(p, r)^{(1/a)}}{m}, \quad (i = 1, \ldots, n)
\]
where \( m^* = m - a\tilde{P}(p, r) \). This system has a nice interpretation. The budget share translation term contains both the commodity specific demographic differences across households, given by \( r_i \), and a measure of deflated expenditures, given by \( m/\tilde{P}^{1/a} \). Similar models, having quantity translation terms vary with deflated expenditures, are discussed in the next section.

Another interpretation comes from putting (21) back into quantity form, which is

\[
D_i(m, r) = D_i^*(m^*, p) + r_i \frac{\tilde{P}^{1/a}}{p_i}, \quad (i = 1, \ldots, n). \tag{22}
\]

In this model, the "committed" quantity of good \( i \) depends on the price of the good, relative to a general price index \( \tilde{P}^{1/a} \), as well as on the demographic characteristics of the family. There is still a sharp division between the necessity levels and the super-numerary expenditures, but now the demographically varying bundle of overheads is no longer fixed. Instead, it has a rudimentary technology of its own.

5. AN APPLICATION: GENERALIZED SCALING AND TRANSLATING OF QUANTITY DEMANDS

This section begins with the quantity equation analogy to Theorem 7. Next, generalized scaling and translating modifying functions that exactly mimic the functional form of ordinary scaling and translating are considered. In terms of the demand equations, these transformations look exactly like committed quantities or equivalent scales would look if these terms could vary with \( p \) and \( m \).

**Theorem 9.** Let \( n = n^* \), and let a modified demand system be given by

\[
D_i(m, p, r) = T_i[D_i^*(m^*, p^*), m, p, r], \quad (i = 1, \ldots, n) \tag{23}
\]

where \( T_i \) is independent of \( D_j^* \) for all \( j \neq i \), for \( i = 1, \ldots, n \). Then \( T_i \) has the form

\[
D_i(m, p, r) = S_i(m, p, r)D_i^*(m^*, p^*) + R_i(m, p, r) \quad (i = 1, \ldots, n) \tag{24}
\]

where, for some modifying functions \( f \) and \( h \),

\[
R_i(m, p, r) = \frac{\partial f(m^*, p, r)}{\partial p_i}, \quad (i = 1, \ldots, n) \tag{25}
\]

\[
S_i(m, p, r) = \frac{h_i(p, r)}{p_i m^*} [m - \sum_{j=1}^{n} p_j R_j(m, p, r)], \quad (i = 1, \ldots, n) \tag{26}
\]

and \( \partial h_i(p, r)/\partial p_j = 0 \) for all \( j \neq i \).

Theorem 9 shows that all modified demand systems that have each demand equation being only a function of \( f, h \), and the corresponding starting demand equation are of the form of generalized scaling and translation.

An example of generalized quantity scaling and translating modifying functions is \( h(p) = p \) and \( f(m^*, p, r) = \tilde{P}^{a} m^{*(1-a)} \) where \( \tilde{P} = \tilde{P}(p, r) \) as in Theorem 8, \( a \) is a constant satisfying \( 0 < a < 1 \), \( r_i \geq 0 \) for \( i = 1, \ldots, n \), and \( \sum_i r_i = 1 \). It is straightforward to show that this \( f \) and \( h \) satisfy Theorems 1, 2, and 3, and that they yield the modified demand system

\[
D_i(m, p, r) = (1 - a)(m/m^*)D_i^*(m^*, p) + ar m/p_i, \quad (i = 1, \ldots, n) \tag{27}
\]

where \( m^* = (m/\tilde{P}^a)^{1/(1-a)} \). In this system, the scaling can be interpreted as depending on a rough measure of deflated expenditures, since \( m/m^* = (m/\tilde{P})^{-a/(1-a)} \). The translation terms are "committed" or base level quantities that increase as \( m/p_i \) increases.
Another similar example is $f(m^*, p, r) = \bar{P}^a m^{(1-a)}$, where $\bar{P}$ is defined as $\bar{P}(p, r) = p' r$, $a$ is a constant satisfying $0 < a < 1$, and $r_i \geq 0$ for $i = 1, \ldots, n$. These are also easily verified to be modifying functions, and they yield the system

$$D_i(m, p, r) = (1 - a)(m/m^*)D_i^*(m^*, p) + ar_i m/\bar{P}, \quad (i = 1, \ldots, n) \tag{28}$$

where $m^* = (m/\bar{P}^a)^{1/(1-a)}$, and $m/m^* = (m/\bar{P})^{-a/(1-a)}$. This system has scales that vary with a measure of deflated expenditures and base level quantities that increase linearly with the deflated expenditure level, $m/\bar{P}$. This increase would be unwarranted if committed quantities were viewed as subsistence level purchases as interpreted by Samuelson (1948) for a linear expenditure system, and by Frisch (1959) when he uses money flexibility as a welfare measure. However, a utility function is a subjective assessment, and committed quantities are only a base level of purchases the consumer will, or perceives he must, make. Paying the maid may be just as much a part of a rich man's committed expenditure as a loaf of bread is in a poor man's. One could even argue that it is precisely the rich consumer's attitudes about standards or living and "necessary" expenditures that drives him to gain the wealth required to support large committed expenditures. In fact, Lewbel (1981) finds high committed expenditures for high income consumers.

The important point here is that models like (27) and (28) have meaningful committed quantities technologies. If other, more (or less) sophisticated household technologies are desired, they too may be found, or be shown not to exist, using the cost function modification technique.

For example, consider ordinary commodity specific translating, which corresponds to transformed demand equations $D_i(m, p) = D_i^*(m - p'r, p) + r_i$ and, as described in Section 3, is given by $f = m^* + p' r$ and $h = p$. The following theorem shows the from modifying functions must take to make transformations identical to those of ordinary translation, except that $r_i$ replaced by a generalized translation term $R_i(m, p, r)$.

**Theorem 10.** Let $n = n^*$. Then modified demands are of the form

$$D_i(m, p, r) = D_i^*[m - \sum_{j=1}^{n^*} p_j R_j(m, p, r), p] + R_i(m, p, r), \quad (i = 1, \ldots, n) \tag{29}$$

if and only if $h(p, r) = p$ and $f(m^*, p, r) = m^* + \alpha(p, r)$ where the function $\alpha$ is positive, homogeneous of degree one in $p$, concave in $p$, and has $\partial \alpha(p, r)/\partial p_i \geq 0$ for $i = 1, \ldots, n$.

An example of demographic translating in the form of equation (29) is given in equation (22). In fact, any modified demand system in the form of equation (24) must also be a transformation of the form of equation (15), since equation (24), when converted to budget share form, is in the form of equation (14). Generalized scaling and translation of quantities is also generalized scaling and translation of budget shares.

Now consider ordinary commodity specific scaling, which corresponds to transformed demand equations $D_i(m, p, r) = r_i D_i^*(m, r_i p_1, \ldots, r_i p_n)$ and, as described in Section 3, is given by $f = m^*$ and $h_i = r_i p_i$. The following theorem shows that, unlike translating, the variables $r_i$ in this demand transformation cannot be replaced by generalized scaling terms $r_i$ that vary with $m$ or $p$.

**Theorem 11.** Let $n = n^*$. Then modified demands of the form

$$D_i(m, p, r) = R_i(m, p, r) D_i^*[m, R_i[m, p, r] p_1, \ldots, R_n[m, p, r] p_n], \quad (i = 1, \ldots, n) \tag{30}$$
do not exist for any modifying functions $f$ and $h$, unless $R_i(m, p, r)$ is independent of $m$ and $p$ for $i = 1, \ldots, n$.

6. SUMMARY AND SUGGESTIONS FOR FUTURE WORK AND APPLICATIONS

Modifying functions provide an easy to use, general procedure for creating new, integrable demand systems out of old ones. The technique encompasses as special cases many commonly used methods for varying demand systems, and may be meaningfully interpreted as describing intermediate good utility production functions. Two stage utility maximization is also a special case of a modified demand system. Modifying functions were applied to the problem of finding new utility functions with desirable properties, such as budget share translating, as well as giving new ways to incorporate demographic variables into demand systems. The search for applications should be continued, exploiting more fully the range of household technologies encompassed by modifying functions. The procedure may also prove useful for providing ways of incorporating other types of variables into demand systems, such as fashions, stocks of durables, and perhaps even errors or expectations in dynamic demand systems.

Modifying functions might also be used for cataloging demand systems. For example, demand systems obtainable from a Cobb Douglas system using different modifying functions include the linear expenditure system, and the homothetic indirect translog demand system. Since demand systems obtainable from a given one can have the given one nested in it, this type of catalog could provide a list of alternatives that a given model's adequacy could be easily tested against, using likelihood ratio type tests.

Another possible application is to suggest modifications to empirically found demand systems that would make them integrable, and in some cases simplify determination of the integrability of newly proposed demand systems.

Finally, modifying functions can be used to endow demand systems with new properties, such as the introduction of committed quantity technologies.

APPENDIX

Proof of Theorem 1. Sufficiency: By equation (1),

$$\sum_{i=1}^{n} \frac{dC}{dp_i} p_i = \frac{\partial f}{\partial m^*} \sum_{j=1}^{n^*} \frac{\partial C^*}{\partial p^*_j} p^*_j \sum_{i=1}^{n} \frac{\partial h_i}{\partial p_i} p_i + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} p_i$$

(A.1)

Substitute equation (2) into equation (A.1), apply the Euler equation to $C^*$ (which is homogenous of degree one in prices) and use equation (3) to reduce the right hand side of equation (A.1) to $f(m^*, p, r)$. By equation (1), $f = C$ so equation (A.1) reduces to the Euler equation showing that $C$ is homogenous. Necessity: Define $\theta_j(p, r)$ for $j = 1, \ldots, n^*$ by

$$\theta_j(p, r) = \sum_{i=1}^{n} \frac{\partial h_j}{\partial p_i} \frac{p_i}{p^*_j}, \quad j = 1, \ldots, n^*$$

and define $\bar{\theta}(m^*, p, r)$ by

$$\bar{\theta}(m^*, p, r) = \left[ 1 - \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} p_i \right] / \left[ \frac{\partial f}{\partial m^*} m^* \right].$$
For $C$ to be homogeneous the equation
\[ \bar{\theta} = \sum_{j=1}^{n^*} \frac{\partial C^*}{\partial p^*} \frac{p_j}{C^*} \theta_j \] (A.2)
must hold for all legitimate cost functions $C^*$, by the Euler equation applied to $C$, and the definitions of $\theta_j$ and $\bar{\theta}$. In particular, equation (A.2) must hold for every cost function of the form $C^* = u p_j$ for $j = 1, \ldots, n^*$. Plugging this cost function for each $j$ into equation (A.2) gives $\bar{\theta} = \theta_j$ for $j = 1, \ldots, n^*$.

**Proof of Theorem 2.** Sufficiency: $f \equiv 0$ implies $C \equiv 0$. $\partial f / \partial m^* > 0$ says $f$ is increasing in $m^*$. So, since $C^*$ is increasing in $u$, $C$ is increasing in $u$. To show that $C$ is nondecreasing in $p$, see
\[ \frac{\partial C}{\partial p_i} = \frac{\partial f}{\partial m^*} \sum_{j=1}^{n^*} \frac{\partial C^*}{\partial p_j^*} \frac{\partial h_j}{\partial p_i} \] (A.3)
for $i = 1, \ldots, n$. The properties of $C^*$ and those required of $f$ and $h$ by the conditions of the theorem make all the right hand side terms of equation (A.3) nonnegative, so $\partial C / \partial p_i \geq 0$ for $i = 1, \ldots, n$. Next, $\partial C / \partial p_i > 0$ is required for at least one price $p_i$. If $\partial f / \partial p_i > 0$ for some $i$, then $\partial C / \partial p_i > 0$ for that $i$. Otherwise, if $\partial f / \partial p_i = 0$ for all $i = 1, \ldots, n$, consider $C$. It is a property of $C$ that $\partial C / \partial p_j > 0$ for some $j$. By the theorem’s statement, if $\partial f / \partial p_i = 0$ (for all $i$), then there exists for this $j$ a function $\tilde{i}(j)$ such that $\partial h_j / \partial p_{i(j)} > 0$. For $i = \tilde{i}(j)$, $\partial C / \partial p_i > 0$.

**Necessity:** This is obvious for the sign of $f$ and for $f$ increasing in $m^*$. For the price derivatives, consider the set of cost functions $C^* = u p_j$ for $j = 1, \ldots, n^*$ and $u > 0$. For these cost functions
\[ \frac{\partial C}{\partial p_i} = \frac{\partial f}{\partial m^*} u \frac{\partial h_i}{\partial p_i} + \frac{\partial f}{\partial p_i}, \] (A.4)
in all $i$, $\partial C / \partial p_i$ must be nonnegative. For arbitrarily small $u$, this implies that $\partial f / \partial p_i \equiv 0$ for all $i$. Rearranging equation (A.3) shows that $\partial C / \partial p_i$ nonnegative if and only if
\[ \frac{\partial f}{\partial m^*} \frac{\partial h_j}{\partial p_i} \equiv - \frac{\partial f}{\partial p_i} u^{-1}, \] (A.5)
in all $i$ and $j$. Given what has been established so far, equation (A.5) with arbitrarily large $u$ shows that $\partial h_j / \partial p_i > 0$ for all $i$ and $j$ is necessary. Finally, $\partial C / \partial p_i$ being strictly positive for some $i$ requires that either $\partial f / \partial p_i > 0$ for some $i$, or $\partial h_j / \partial p_i > 0$ for some $i$.

**Proof of Theorem 3.** Sufficiency: The function $C(u, p)$ is concave in the vector $p$ if, at $t = 0$, $\frac{\partial^2 C(u, p + vt)}{\partial t^2} \leq 0$ for all $n$ vectors $v$ (see Hardy, Littlewood and Polya (1952)). Straightforward matrix differentiation shows that
\[ \frac{\partial^2 C(u, p + vt)}{\partial t^2} \bigg|_{t=0} = \sum_{i=1}^{n^*} \left( \left[ \frac{\partial C^*}{\partial p_i^*} \right]^2 + \left[ \frac{\partial C^*}{\partial p_i^*} \right] B_1(i, \cdot) + \left[ \frac{\partial C^*}{\partial p_i^*} \right] B_2(i, \cdot) + B_3(\cdot) \right) \]
\[ + \left[ \frac{\partial f}{\partial m^*} v \frac{\partial h}{\partial p} \frac{\partial^2 C^*}{\partial p^* \partial p^*} \frac{\partial h^t}{\partial p} \right] \] (A.6)
where $B_1$, $B_2$, and $B_3$ are given by equations (4), (5), and (6), respectively. The terms involving $B_1$ and $B_3$, and the final term in equation (A.6) are all nonpositive because $C^*$ is concave and $\partial f / \partial m^* \equiv 0$. The constraint $B_2 \equiv 2 \sqrt{B_1 B_3}$ is necessary and sufficient to
ensure that the equation $X B_1 + X B_2 + B_3$ has no nonnegative real roots $X$. Therefore, with $B_1$ nonpositive, $X B_1 + X B_2 + B_3 \leq 0$ for all $X \geq 0$. With $X = \partial C^* / \partial p_i^*$, this shows that equation (A.6) is a sum over all $i$ of nonpositive terms, so $C$ is concave.

Necessity: For all legitimate $C^*$, $\partial C(u, p + vt) / \partial t^2 |_{t=0} \leq 0$ must be nonpositive. In particular, it must be nonpositive for cost functions $C^* = u p_i^*$ for $u > 0$ and $i = 1, \ldots, n^*$. For this set of cost functions, equation (A.6) reduces to

$$\frac{\partial^2 C(u, p + vt)}{\partial t^2} \bigg|_{t=0} = u^2 B_1(i, \cdot) + u B_2(i, \cdot) + B_3(\cdot).$$

For arbitrarily small $u$, $\partial^2 C / \partial t^2 |_{t=0} \leq 0$ requires $B_3 \leq 0$. For arbitrarily large $u$, $\partial^2 C / \partial t^2 |_{t=0} \leq 0$ requires $B_1(i, \cdot) \leq 0$. As before, with $B_1 \leq 0$, the constraint $\sum B_i \leq 0$ is necessary for $\partial^2 C / \partial t^2 |_{t=0} \leq 0$ to hold for all values of $u > 0$. 

Proof of Theorem 4. The indirect utility function $V$ is the inverse of $C$, so $m = C[V(m, p, r), p, r]$. Using equation (1) as the definition of $C$, this equation for $m$ becomes

$$m = f[C^*(V(m, p, r), p, r), p, r]. \quad (A.7)$$

Now, $V^*(m^*, p^*) = V^*[F(m, p, r), p^*]$ by the definition of $F$. Substitute equation (A.7) for $m$ into this equation. After cancelling out $C^*$ of $V^*$, and $f$ of $F$, this reduces to $V^*(m^*, p^*) = V(m^*, p, r)$ for the demand system, equation (8) is obtained by differentiating equation (1) with respect to $p_i$, and substituting in $H_i^* = \partial C_i / \partial p_i$. Marshallian demands are obtained by substituting $V^*(m^*, p^*)$ for $u$ on the right side of equation (8), and $V(m, p)$ for $u$ on the left side. Equation (10) comes from equation (9) by replacing $D_i$ with $(m/p) W_i$ and $D_i^*$ by $(m^*/p^*_i) W_i^*$, and solving for $W_i$.

Proof of Theorem 5. Let $p^* = h(p)$ be a constant vector for all $p$, and let $f(m^*, p, r) = C(m^*, p, r)$. It is straightforward to show that $h$ and $f$ are modifying functions, given the properties of any legitimate cost function $C$. Let $C^{**}(u, p, r)$ be the modified demand system corresponding to $f$ and $h$ applied to $C^*$. Then, by construction, $C^{**}(u, p, r) = C[C^*(u, p^*), p, r]$. Since $p^*$ is constant, $C^*(u, p, r)$ is a nondecreasing monotonic function of one variable, $u$. Therefore, $C^{**}(u, p, r)$ describes the same preferences as $C(u, p, r)$.

Proof of Theorem 6. Start with equation (9), and replace $D_i$ with $q_i D_i^*$ with $q_i^*$, and $m^*$ with $F(m, p, r)$. Next, by homogeneity, replace $m$ with 1 and $p_i$ by $p_i/m$, then observe that by definition $p_i/m = \phi_i(q, r)$. The result, after rearranging terms, is equation (11). Since $q$ and $q^*$ both exist, equation (11) must have a solution for $q^*$ given $q$. When rank $(\partial h / \partial p) = n^*$, $\partial h / \partial p$ has a left inverse, which makes the solution for $q^*$ given $q$ be unique.

Proof of Theorem 7. Nesting equation (14) into the form of equation (10) shows that

$$T_i(W_i^*, m, p, r) = \frac{\partial f}{\partial m^*} m^* \frac{\partial h_i}{\partial p_i} p_i W_i^* + \frac{\partial f}{\partial p_i} p_i W_i^*, \quad (i = 1, \ldots, n) \quad (A.8)$$

So $R_i = (\partial f / \partial p_i) p_i/m$, and $\partial h_i / \partial p_j = 0$ for $j \neq i$. By equation (2) it then follows that $(\partial h_i / \partial p_i)(p_i / p_i^*) = \theta(p, r)$ for each $i$. Next, equation (3) makes the coefficient of $W_i^*$ in equation (A.8) reduce to $1 - \sum (\partial f / \partial p_i)(p_i / m)$, which completes the proof.

Proof of Theorem 8. By Theorem 7, $r_{i+n}$ in equation (16) must equal $s(r)$ for $i = 1, \ldots, n$, and $r_i$ must equal $(\partial f / \partial p_i) p_i/m$. Since $f = m$ and this expression for $r_i$ is
the right hand side of equation (3) gives 

\[ 1 - \sum \gamma_i = (\partial f / \partial m^*)(m^*/m)\theta(p, r) \]

which by the above expressions for \( s \) and \( f \) reduces to \( s(r) = (\partial A / \partial m^*)(m^*/A)\theta(p, r) \). \( s \) and \( \theta \) are independent of \( m^* \), and \( A \) is independent of \( p \). So this expression for \( s(r) \) implies that \( \theta(p, r) = \alpha(r) \) for some function \( \alpha \), and either \( A(m^*, r) = \beta(r)m^*[s(r)/\alpha(r)] \) or \( A(m^*, r) = \beta(r) \) and \( \alpha(r) = 0 \) for some function \( \beta \). These expressions for \( \theta \), equation (2), and the fact from Theorem 7 that \( \partial h_i / \partial p_i = 0 \) for \( i \neq j \) together imply that \( (\partial h_i / \partial p_i)p_i/p_i^* = \alpha(r) \). Since \( \alpha \) is independent of \( p \), this expression for \( \alpha(r) \) means that \( h_i \) has the form of equation (17) for some nonnegative function \( \gamma(r) \). Also, the above expressions for \( A \) and \( f \) say that either \( f \) is given by equation (18), or \( \alpha(r) = 0 \) and \( f(m^*, p, r) = \beta(r) \). These expressions for \( f \) and \( h \) satisfy Theorem 1. By theorem 2, \( \partial f / \partial m^* > 0 \) which rules out \( f = \beta(r) \). So it is now shown that \( f \) and \( h \) must have the form given by equations (17) and (18). The restrictions from Theorem 2 that \( f \geq 0, \partial f / \partial m^* > 0, \partial f / \partial p_i \geq 0 \) and \( \partial h_i / \partial p_i = 0 \) for \( i \neq j \) imply that \( \beta(r) > 0, s(r)/\alpha(r) > 0, r_i \geq 0 \) and \( \alpha(r) \gamma(r) \geq 0 \). Combined with \( \gamma(r) > 0 \) and the expression for \( s(r) \), these inequalities show that \( \alpha(r) > 0 \) and \( s(r) > 0 \). The last restriction in Theorem 2 is satisfied by \( \gamma(r) > 0 \), which makes \( \partial h_i / \partial p_i > 0 \) for all \( i \). By Theorem 3, \( B_1 \leq 0 \) is obtained by \( s(r) \leq \alpha(r) \), which makes \( f \) concave in \( m^* \). \( B_1 \leq 0 \) comes from \( f \) being concave in \( p \), which is obtained by \( 0 \leq s(r) \leq 1 \). The remainder of this proof is devoted to showing that the upper bound given for \( \alpha(r) \) in this theorem is necessary and sufficient to have \( B_2 \leq 2 \sqrt{B_1 B_2} \). Straightforward differentiation shows that

\[ B_1(i, \cdot) = \left[ \frac{s}{\alpha} \right] \left[ \frac{s - 1}{\alpha^2} \right] \frac{s}{m^*} (s h_i/v_i)^2, \quad (i = 1, \ldots, n) \]

\[ B_2(i, \cdot) = \left[ \frac{s}{\alpha} \right] \frac{s}{m^*} \alpha(\alpha - 1) h_i v_i^2 / p_i^2, \quad (i = 1, \ldots, n) \]

\[ B_3(\cdot) = \frac{1}{n} \sum [\Sigma_j [r_i f(v_i/p_i)(v_j/p_j)] - r_i f v_i^2/p_i^2]. \]

Now, \( B_2 \leq 2 \sqrt{B_1 B_3} \) implies either \( B_2 \leq 0 \) or \( B_2^2 \leq 4 B_1 B_3 \) for all \( v \). \( B_2 \leq 0 \) holds if \( \alpha \leq 1 \), and \( B_2^2 \leq 4 B_1 B_3 \) holds for a given \( i \) if \( v_i = 0 \). For \( v_i \neq 0 \), the expression \( B_2^2 \leq 4 B_1 B_3 \) is reducible (with \( p \)'s absorbed into \( v \)'s) to

\[ \frac{s(\alpha - 1)^2 n}{4(\alpha - s)} \leq \frac{1}{v_i^{-2}}[\Sigma_j r_i v_j^2] - \Sigma_j r_i v_j^2 \]

(A.9)

which, if \( \alpha > 1 \), must hold for all \( i \), and all values of \( v \) except \( v_i = 0 \). Let \( \tilde{v}(r, v) \) equal the right hand side of equation (A.9). It is necessary and sufficient that equation (A.9) hold for the \( \min \tilde{v}(r, v) \). The function \( \tilde{v}(r, v) \) is homogeneous of degree zero in \( v \), so one normalization can be made without affecting the value of \( \tilde{v} \). Choose the normalization \( v_i = 1 \). The first order conditions for minimizing this function over all \( v_i \) for \( j \neq i \) have the unique solution \( v_i = r_i/(s + r_i) \) for all \( j \neq i \), which makes \( \tilde{v}(r, v) = r_i s/(s + r_i) \). Evaluating \( \tilde{v} \) for alternate values of \( v \) shows that this solution is in fact a minimum. Since (A.9) must hold for all \( i \) it is required that \( s(\alpha - 1)^2 n / 4(\alpha - s) \leq \min [r_i s/(s + r_i)] \), which is equivalent to \( (\alpha - 1)^2 s - 2(\alpha - s) \delta(r) \leq 0 \) where \( \delta(r) \) is as defined in the theorem. This inequality is a convex quadratic in \( \alpha \), and therefore is satisfied when \( \alpha \) lies between the roots of this quadratic. These roots are \( 1 + \delta(r) \pm \sqrt{\delta(r)^2 + [2 - 2s(r)] \delta(r)} \). So, \( B_2 \leq 2 \sqrt{B_1 B_3} \) when \( \alpha(r) \) lies between these roots or when \( \alpha(r) \leq 1 \). The smaller of these roots is less
than one, so the lower bound for $\alpha(r)$ is $s(r)$, as proven earlier, and the upper bound is the greater root above.

Proof of Theorem 9. Nest equation (23) into equation (9) to get equation (24). Transforming equation (24) into budget share form yields $W_i = (p m^*/p_i^* m) S_i W^* + R_iP_i/m$. Applying Theorem 7 to this equation proves the remainder of this theorem.

Proof of Theorem 10. Nesting equation (29) into equation (9) shows that $h_i = p_i, \partial f/\partial p_i = R_i, \partial f/\partial m^* = 1$, and $m^* = m - \sum_i R_i p_i$. Substitute $f(m^*, p, r)$ for $m$ in this last equation for $m^*$, and take the derivative with respect to $m^*$ to yield $1 = \partial f/\partial m^* - \sum_i (\partial R_i/\partial m)(\partial f/\partial m^*)p_i$. Substitute $\partial f/\partial m^* = 1$ to get $\partial (\sum_i R_i p_i)/\partial m = 0$, so $\sum_i R_i p_i$ is independent of $m$. Define the function $\alpha(p, r)$ by $\alpha(p, r) = \sum_i R_i p_i$. Then, the above equation above for $m^*$ shows that $f = m^* + \alpha(p, r)$. Next, by $\partial f/\partial p_i = R_i$ and the definition of $\alpha$ get $f = m^* + \sum_i (\partial f/\partial p_i)p_i$. This reduces to $\alpha(p, r) = \sum_i [\partial \alpha(p, r)/\partial p_i]p_i$, which is the Euler equation showing that $\alpha(p, r)$ is homogeneous of degree one in $p$. Substituting $f$ and $h$ above into equations (2) and (3) shows that theorem 1 holds, with $\theta(p, r) = 1$. In addition to the relationships given at the beginning of this proof, Theorem 2 requires $\partial f/\partial p_i \equiv 0$, which means $\partial \alpha/\partial p_i \equiv 0$, and theorem 2 requires $f \equiv 0$, so $\alpha \equiv 0$. In theorem 3, $B_1 = 0$ because $\partial^2 f/\partial m^* = 0$, and $B_2 = 0$ because $\partial^2 f/\partial p_i \partial p_k = 0$ for all $i, j$, and $k$. Finally, $B_3 \leq 0$ requires $f$ to be concave in $p$, which means $\alpha(p, r)$ must be concave in $p$.

Proof of Theorem 11. Nesting equation (30) into equation (9) shows first that $m^* = m = f(m^*, p, r)$ so $f$ is independent of $p$ and $r$ and is the identity function with respect to $m^*$, and second that $h_i(p, r) = R_i(m, p, r)p_i$ which shows that $R_i$ is independent of $m$. From Theorem 4, $\partial h_i/\partial p_i = R_i(m, p, r)$ and $\partial h_i/\partial p_j = 0$ for $j \neq 1$, so $\partial (R_i p_i)/\partial p_i = R_i$ and $\partial (R_i p_i)/\partial p_j = 0$ for $j \neq 1$, and therefore $R_i$ is independent of $p$.

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NOTES

1. Equivalent scales originated with Engel (1893). Commodity specific demographic effects may be traced back to Sydenstricker and King (1921), although they weren’t made popular until Prais and Houthakker (1955). Barten (1964), Gorman (1976), Muellbauer (1974), and Pollak and Wales (1981) among others, have formulated or catalogued equivalent scales models and variants of them that are consistent with utility theory. I will use Pollak and Wales’ syntax of scaling and translating to refer to these models.

2. For example, a model as simple as the Cobb-Douglas with demographically varying constant budget shares cannot be derived from any demand system using translating or Barten style scaling.

3. Lau, Lin, and Yotopolous (1978) briefly consider the addition of demographic parameters to an indirect utility function, which is equivalent to adding them to the cost function, but they quickly proceed to a specific demand system for their application, without performing an analysis like the one given here.

4. For an elementary discussion of cost functions and these properties, see, for example, Deaton and Huwicz and Uzawa (1971), and Blackorby, Primont, and Russell (1978).

5. For notational convenience, all gradient vectors, Jacobean matrices, and Hessian matrices are represented as derivatives of by vectors.

6. I wish to thank an anonymous referee for calling this issue to my attention.

7. Exceptions are models that are not universally applicable to all demand systems, like the Prais and Houthakker (1955) model. As Muellbauer (1977) and Pollak and Wales (1981) observe, this demographic transformation may not be performed on all demand systems. Therefore, it cannot be encompassed by the modifying functions technique.
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