On nonparametric demand analysis

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Abstract

Building on the work of Afriat and Varian, a nonparametric analysis of consumption behavior is presented. Inner-bound and outer-bound representations of preferences are obtained that are consistent with both the data and consumer theory. It is shown how a priori information on demand response can be incorporated in the analysis. The approach generates estimates of price and income elasticities of demand. An application to U.S. consumption data illustrates the usefulness of the method for predicting demand behavior as well as for welfare analysis.

JEL classification: D1; C6; D6

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I. Introduction

Nonparametric methods have been proposed for the analysis of consumer behavior (Afriat, 1967, 1987; Diewert, 1973; Diewert and Parkan, 1985; Landsburg, 1981; Varian, 1982, 1983, 1985; Chiappori and Rochet, 1987). These nonparametric methods generate tests of observed data for consistency with consumer theory without ad hoc specification of a functional form for preferences (direct or indirect utility, or expenditure functions) and/or demand functions. The use of these methods as a complement to more traditional parametric demand

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analysis is evident. For example, the nonparametric approach can be useful to test for maintained hypotheses (such as the separability implicitly imposed by a commodity grouping) prior to parametric estimation (e.g., Barnhart and Whitney, 1988; McMillan and Amoako-Tuffour, 1988; Swofford and Whitney, 1987; Sakong and Hayes, 1993). It is also useful in conducting welfare analysis (e.g., Varian, 1982; Manser and McDonald, 1988).

Varian (1982) discussed nonparametric techniques of revealed preference analysis for the following: (1) test a finite amount of data for consistency with utility maximization model; (2) construct a nicely behaved utility function capable of rationalizing a finite demand data; (3) compare previously unobserved consumption bundles and budgets with respect to their ordinal ranking; (4) compute bounds on the direct and indirect compensation functions; and (5) compute estimates of the direct and indirect demand correspondences consistent with previously observed demand data. While the use of nonparametric techniques for testing utility maximization and/or separability is now fairly standard, the use of these methods for the analysis of Varian’s (5) above has not enjoyed similar adoption. In particular, nonparametric estimation of demand response (e.g., price and income elasticities) has apparently not appeared in the literature. This suggests a need to explore further the empirical linkages between nonparametric methods and demand analysis. The objective of this paper is to investigate the usefulness of the nonparametric approach for applied demand and welfare analysis.

The manuscript contributes to previous literature on the economics of consumer behavior in several ways. First, it presents a method to estimate nonparametrically price and income elasticities of demand from observed consumption data. Second, it shows how to incorporate a priori information about consumer behavior in nonparametric demand and welfare analysis. Finally, the paper assesses some of the empirical implications of the non-unique representations of consumer preferences that are consistent with observable behavior.

The paper is organized as follows. A review of some key results obtained by Afriat (1967, 1987) and Varian (1982) is presented in Section 2. The intent of this section is to stress the seminal nature of the Afriat–Varian methodology and to make it more accessible for empirical research. The use of these nonparametric methods in empirical demand response analysis is presented in Section 3. We show how to estimate nonparametrically price and income elasticities of demand, and how to incorporate a priori information in nonparametric analysis (e.g., imposing the restriction that some income elasticities are non-negative). Section 4 focuses on welfare analysis and nonparametric estimation of upper and lower bounds on cost-of-living and standard-of-living indexes. Section 5 illustrates the approach with an application to aggregate U.S. consumption data, 1953–1992. The analysis generates nonparametric estimates of Marshallian price and income elasticities associated with the inner and outer bound representation of consumer preferences. It also provides a basis for estimating bounds on cost-of-living and standard-of-living indexes. Concluding remarks are presented in Section 6.
2. The nonparametric approach

This section summarizes some key results presented in Afriat and Varian. The nonparametric approach to demand analysis consists in analyzing a finite body of data without ad hoc specification of functional forms for utility or demand functions (see Afriat, 1967, 1987; Varian, 1982, 1983). Assume that household behavior is consistent with the maximization of a utility function \( u(x) \) with respect to the \((n \times 1)\) consumption vector \( x \), subject to the budget constraint \( p' x \leq l \), where \( p > 0 \) is a \((n \times 1)\) vector of commodity prices and \( l > 0 \) is household income. Defining normalized prices as the \((n \times 1)\) vector \( v = p/l \), this can be written as

\[
x^*(v) = \arg \max \{ u(x) : v' x \leq 1, \, x \geq 0 \}, \tag{1}
\]

where \( x^*(v) \) is the Marshallian demand correspondence.

Assume that a household is observed making consumption decisions \( \tau \) times. Thus, there are \( \tau \) (finite) observations on consumption decisions \((x_i)\), corresponding prices \((p_i)\) as well as income \((I_i)\), \( i = 1, \ldots, \tau \). Let \( T = \{1, \ldots, \tau\} \) denote the set of these observations. We define economic rationality for consumption decisions in terms of Eq. (1). Thus, we say that a utility function \( u(x) \) rationalizes the data \( \{(x_i, v_i) : t \in T\} \) if \( x_i \in x^*(v_i) \), \( t \in T \). Some key linkages between observable behavior and consumer theory (as given by Eq. (1)) are presented next (Afriat, 1967, 1987; Varian, 1982, 1983):

**Afriat’s Theorem.** The following conditions are equivalent:
1. There exists a non-satiated utility function that rationalizes the data according to (1).
2. The data satisfy the Generalized Axiom of Revealed Preferences (GARP).
3. For all \( s, t \in T \), there exist numbers \( U_s, \lambda_t \) that satisfy the Afriat inequalities:
   \[
   \lambda_t > 0, \tag{2a}
   \]
   \[
   U_s \leq U_t + \lambda_t [v'_t x_s - 1]. \tag{2b}
   \]
4. There exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data according to (1).

Eq. (2a) and Eq. (2b) present necessary and sufficient conditions for the data \( \{(x_s, v_t) : t \in T\} \) to be consistent with utility maximization (1), where \( U = (U_1, U_2, \ldots, U_T) \) can be interpreted as a \((\tau \times 1)\) vector of utility levels and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_T) \) as a \((\tau \times 1)\) vector of marginal utilities of income (Afriat, 1967, 1987; Varian, 1982, 1983). The non-satiating condition associated with \( \lambda_t > 0 \) implies that the budget constraint is necessarily binding at the optimum: \( v'_t x_t = 1 \) for all \( t \in T \). Expressions (2a) and (2b) can be evaluated with combinatorial.

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1 As noted in the introduction, many of the results presented in this section are not new. The intent here is to present the Afriat (1987) results in a form more readily accessible to economists. This section also provides the basis for the developments presented in subsequent sections.
rial methods (e.g., via Marshall’s algorithm as proposed by Varian (1982, 1983)) or using linear programming techniques (Diewert, 1973; Diewert and Parkan, 1985; Varian, 1982, pp. 946–947 and footnote 3).

Afriat’s theorem establishes empirically tractable conditions for the existence of a preference function that can rationalize observable demand behavior. In particular, under non-satiation, it implies that the existence of a solution to Eqs. (2a) and (2b) for the U’s and λ’s is necessary and sufficient for the observed consumer behavior to be consistent with utility maximization in Eq. (1).

Perhaps more importantly, Afriat’s theorem provides a basis for recovering a utility function representing consumer preferences that is fully consistent with the data and the theory. This can be of considerable interest since the utility function provides all the information necessary to conduct either a predictive analysis (e.g., the responsiveness of demand to prices and income) or a normative analysis of consumer behavior (e.g., the evaluation of household welfare). The problem is that the utility function that can be recovered from the data is not unique. There exists a whole family of utility functions that are consistent with observable behavior and the utility maximization hypothesis (1). As shown by Afriat (1987), this family can be bounded by two particular representations of the utility function, \( w^*(x, \cdot) \) and \( w^*(\cdot, x) \). These functions and their properties are discussed next. Our treatment follows Afriat (1987). For the sake of completeness, all proofs are presented in the Appendix.

2.1. The function \( w^*(x, U) \)

For a given set of \( U \) and \( \lambda \) satisfying Eqs. (2a) and (2b), define the function \( w^*(x, U) \) as follows (Afriat, 1987, p. 128):

\[
\begin{align*}
\hat{w}^*(x, U) &= \max_{\theta} \left\{ \sum_{i \in T} U_i \theta_i : \sum_{i \in T} x_i \theta_i \leq x; \sum_{i \in T} \theta_i = 1; \theta_i \geq 0, i \in T \right\},
\end{align*}
\]

where \( \theta_i \) is a scalar, and \( \theta = \{ \theta_i, i \in T \} \). By construction, \( w^*(x, U) \) in Eq. (3) is a non-decreasing and concave function of \( x \). Note that the linear programming problem (3) can be alternatively expressed in terms of its dual. To see this, denote \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)' \) as the \((n \times 1)\) vector of Lagrange multipliers associated with the constraints \( \sum_{i \in T} x_i \theta_i \leq x \) in Eq. (3), and \( \beta \) as the Lagrange multiplier associated with the constraint \( \sum_{i \in T} \theta_i = 1 \) in Eq. (3). Then, the equivalent dual representation of the primal linear programming problem (3) is

\[
\begin{align*}
\hat{w}^*(x, U) &= \max_{\alpha, \beta} \min_{\theta} \left\{ \sum_{i \in T} U_i \theta_i + \alpha' \left( x - \sum_{i \in T} x_i \theta_i \right) + \beta \left( 1 - \sum_{i \in T} \theta_i \right) : \theta_i \geq 0, \right. \\
& \left. \quad t \in T; \alpha \geq 0 \right\} \\
& = \min_{\alpha, \beta} \left\{ x' \alpha + \beta : x' \alpha + \beta \geq U_i, t \in T; \alpha \geq 0 \right\},
\end{align*}
\]

\[ (3') \]
where $\theta_t$ is the Lagrange multiplier associated with the constraint $[x_t \alpha + \beta \geq U_t]$ in Eq. (3'), $t \in T$.

**Lemma 1.** $w^i(x, U)$ is a representation of consumer preferences which rationalizes the data $(x_t, v_t), t \in T$.

Lemma 1 states that $w^i(x, U)$ in Eq. (3) provides a representation of consumer preferences that is consistent with both optimizing behavior (1) and the data $\{(x_t, v_t); t \in T\}$ (Afriat, 1987, p. 127). As discussed next, this representation is not unique.

### 2.2. The function $w^\alpha(x, U, \lambda)$

For a given set of $U$ and $\lambda$ satisfying Eq. (2a) and Eq. (2b), define the function $w^\alpha(x, U, \lambda)$ as follows (Afriat, 1987, p. 127):

$$w^\alpha(x, U, \lambda) = \min_t \left[ U_t + \lambda_t(v_t x - 1), t \in T \right]. \quad (4)$$

By construction, $w^\alpha(x, U, \lambda)$ is an increasing and concave function of $x$. Note that it can be alternatively expressed as the following primal linear programming problem:

$$w^\alpha(x, U, \lambda) = \max_{\gamma} \left[ \gamma; \gamma \leq U_t + \lambda_t(v_t x - 1), t \in T \right]. \quad (4')$$

**Lemma 2.** $w^\alpha(x, U, \lambda)$ is a representation of consumer preferences which rationalizes the data $(x_t, v_t), t \in T$.

Lemma 2 indicates that $w^\alpha(x, U, \lambda)$ provides a representation of consumer preferences that is consistent with both optimizing behavior (1) and the data $\{(x_t, v_t); t \in T\}$ (Afriat, 1987, p. 127). Thus, we now have two possible representations of consumer preferences: $w^i(x, U)$ and $w^\alpha(x, U, \lambda)$. These two functions are of special interest because, for a given $(U, \lambda)$ satisfying Eqs. (2a) and (2b), they provide nonparametric bounds on the representations of the utility function $u(x)$ in Eq. (1).

### 2.3. Utility bounds

From Afriat's theorem (part d), we know that we can always choose to represent the utility function by a concave, monotonic, continuous, non-satiated function. For this purpose, denote a general utility function by $w(x)$, which is any non-decreasing, concave, non-satiated function. We present next Afriat's result stating that $w^i(x, U)$ and $w^\alpha(x, U, \lambda)$ provide bounds on the utility function $w(x)$ (Afriat, 1987, p. 141).
Proposition 1. For a given set of $U$ and $\lambda$ satisfying Eq. (2a)Eq. (2b), the functions $w'(x, U)$ and $w''(x, U, \lambda)$ provide, respectively, the inner bound and the outer bound representation of consumer preferences:

$$w'(x, U) \leq w(x) \leq w''(x, U, \lambda),$$

(5)

where $w(x)$ is any concave, non-satiated utility function rationalizing the data \{(x_t, v_t), t \in T\} according to Eq. (1) and satisfying $w(x_t) = U_t$ for all $t \in T$.

Proposition 1 shows that the two functions $w'(x, U)$ and $w''(x, U, \lambda)$ provide the tightest bounds on all possible concave and non-satiated utility representations of $u(x)$ in Eq. (1), $w'(x, U)$ giving the inner bound while $w''(x, U, \lambda)$ gives the outer bound. However, it should be emphasized that these bounds are obtained for given values of $U$ and $\lambda$ satisfying the Afriat inequalities (2a) and (2b). This indicates that the bounds derived in Eq. (5) are conditional on $U$ and $\lambda$. In general, the Afriat inequalities (2a) and (2b) may have more than one solution for $U$ and $\lambda$. This would imply that there is more than one value of $(U, \lambda)$ that is available to evaluate the conditional bounds in Eq. (5). And in general, we expect these bounds to be affected by the value of $(U, \lambda)$. This suggests a need to evaluate unconditional bounds for $w(x)$, i.e., bounds that do not depend on the values taken by $(U, \lambda)$.

For this purpose, note that the concave function $w(x)$ is defined only up to a positive linear transformation. This suggests that, without a loss of generality, the Afriat inequalities (2a) and (2b) can be alternatively written as

$$\lambda, \geq \epsilon, \quad t \in T,$$

(6a)

$$U_t \leq U_t + \lambda_t [v_t x_t - 1], \quad s, t \in T,$$

(6b)

$$U_b = 0,$$

(6c)

$$-M \leq U_t \leq M, \quad t \in T,$$

(6d)

where $\epsilon$ is a small positive scalar, $U_b$ is the utility level for some base observation $b \in T$, and $M$ is some large, positive, finite scalar. Eqs. (6a) and (6b) correspond to Eqs. (2a) and (2b). Eq. (6c) is a normalization which arbitrarily fixes the utility level for the base observation $b$. Finally, expression (6d) is used simply to insure that the utility levels $U_t$ remain bounded. In the rest of the paper, we will use Eqs. (6a), (6b), (6c) and (6d) as an equivalent means of imposing the Afriat inequalities (2a) and (2b).

This provides a basis for defining unconditional bounds at point $x$ as follows:

$$W_t(x) = \operatorname{Min}_{U, \lambda} \left[ w'(x, U) : \text{subject to Eq. (6)} \right]$$

(7a)
and

$$W^o(x) = \max_{U, \lambda} \left[ w^o(x, U, \lambda) : \text{subject to Eq. (6)} \right].$$

(7b)

Combining Eqs. (7a) and (7b) with Proposition 1 gives the following result:

**Proposition 2.** Given Eq. (6), the functions $W^i(x)$ in Eq. (7a) and $W^o(x)$ in Eq. (7b) provide, respectively, the unconditional inner bound and outer bound representations of consumer preferences at point $x$:

$$W^i(x) \leq W(x) \leq W^o(x).$$

(8)

Proposition 2 gives the widest possible range for $W(x)$ to provide a representation of consumer preferences consistent with utility maximization and observable behavior. It shows that $W^i(x)$ and $W^o(x)$ provide inner and outer bounds on utility consistent with the data. The usefulness of these bounds in the economic analysis of consumer behavior is discussed next.

### 3. Demand analysis

The unconditional bounds $W^i(x)$ and $W^o(x)$ defined in Eqs. (7a) and (7b) provide a basis for conducting empirical demand analysis. This section focuses on the use of the nonparametric approach in predicting consumer behavior and in conducting sensitivity analysis of demand to changing prices and income. For this purpose, consider the classical utility maximization problem:

$$\max_x \{ W^o(x) : x^+x \leq 1, x \geq 0 \},$$

(9)

where $W^o(x)$ is a particular representation of $u(x)$ in Eq. (1). Let $x^o(v)$ denote the solution of the above optimization problem. Then, $x^o(v)$ is the Marshallian demand correspondence representing the quantity consumed under relative price $v$. Hence, when evaluated at different relative prices, Eq. (9) provides a basis for measuring the anticipated effects of changes in relative prices. The empirical issue then is the evaluation of Eq. (9).

Given Eq. (8), we have two obvious candidates for $W^o(x)$: the unconditional inner bound $W^i(x)$ in Eq. (7a) and the unconditional outer bound $W^o(x)$ in Eq. (7b). First, consider the inner-bound representation $W^i(x)$. Given $W^o(x) = W^i(x)$ and using Eq. (7a) and Eq. (3'), expression (9) takes the form

$$\max_x \left[ W^i(x) : x^+x \leq 1, x \geq 0 \right]$$

$$= \max_{x} \min_{U, \lambda} \left[ w^i(x, U) : \text{Eq. (6)} ; x^+x \leq 1, x \geq 0 \right]$$

$$= \max_{x} \min_{U, \lambda, \alpha, \beta} \left[ x^+x + \beta : x^+x + \beta \geq U_i, t \in T, \alpha \geq 0 ; \text{Eq. (6)} ; \right]$$

$$\quad x^+x \leq 1, x \geq 0 \right].$$

(10)
Denote by $\zeta$ the Lagrange multiplier associated with the budget constraint $(v'x \leq 1)$ in Eq. (10). Noting the existence of a saddle-point to the corresponding Lagrangean $[x'\alpha + \beta + \zeta (1 - v'x)]$, expression (10) can be alternatively expressed using the following dual formulation:

$$\max_x \left[ W'(x) : v'x \leq 1 ; x \geq 0 \right]$$

$$= \max_x \min_{U, \lambda, \alpha, \beta, \zeta} \left[ x'\alpha + \beta + \zeta (1 - v'x) : x'\alpha + \beta \geq U_t, t \in T ; \alpha \geq 0 ; \right.$$

$$\left. \text{Eq.}(6) ; x \geq 0 ; \zeta \geq 0 \right] = \min_{U, \lambda, \alpha, \beta, \zeta} \left[ \beta + \zeta : \alpha - \zeta v \leq 0 ; x'\alpha + \beta \geq U_t, \right.$$ 

$$\left. t \in T ; \alpha \geq 0 ; \text{Eq.}(6) ; \zeta \geq 0 \right].$$

(11)

where $x$ is the $(n \times 1)$ vector of Lagrange multipliers associated with the constraint $(\alpha - \zeta v \leq 0)$ in Eq. (11). Denote by $U^t$, $\lambda^t$ and $x^t$ the solution of Eq. (11). Eq. (11) constitutes a standard linear programming problem that provides a convenient basis to solve for the optimal behavior associated with $W'(x)$, the unconditional inner-bound representation of preferences, under the economic conditions given by the relative price vector $v$.

Second, consider the outer-bound representation $W^o(x)$. Given $W^o(x) = W^o(x)$ and using Eq. (7b) and Eq. (4'), expression (9) takes the form

$$\max_x \left[ W^o(x) : v'x \leq 1 ; x \geq 0 \right]$$

$$= \max_{x, U, \lambda} \left[ w^o(x, U, \lambda) : \text{Eq.}(6) ; v'x \leq 1 ; x \geq 0 \right]$$

$$= \max_{x, U, \lambda, y} \left[ y : y \leq U_t + \lambda_t (v'_t x - 1) , t \in T ; \text{Eq.}(6) ; v'x \leq 1 ; x \geq 0 \right].$$

(12)

Let $x^o$, $U^o$ and $\lambda^o$ denote the solution of the optimization problem (12). Eq. (12) constitutes a non-linear programming problem that provides a convenient basis to solve for the optimal behavior associated with $W^o(x)$, the unconditional outer-bound representation of preferences, under the economic conditions given by the relative price vector $v$.

3.1. Imposing a priori information

We have just shown that Eqs. (11) and (12) provide a basis for predicting consumption behavior. However, there are often situations where the investigator has a priori information about the range within which predicted consumption is expected to lie. This a priori information can be subjective and/or based on empirical evidence from previous studies. For example, it may be reasonable to avoid extrapolating behavior outside the range of the data. This would imply that predicted behavior is restricted to lie within the range of the data. Also, the investigator may know on a priori grounds that some goods are non-inferior, with
non-negative income elasticity. In this context, starting from some base situation \((x_b, v_b), b \in T\), an increase (decrease) in income would imply that the demand for these commodities could not decrease (increase). We consider here the incorporation of such a priori behavioral restrictions in nonparametric demand analysis.

To evaluate demand elasticities, we consider a number of price/income scenarios denoted by \(u_k, k = 1, 2, \ldots, k\). Let \(K = \{1, 2, \ldots, k\}\) denote the set of all these scenarios. For each scenario \(u_k\), we want to estimate the corresponding demand \(x_k, k \in K\). In this context, using \((x_k, u_k)\) as a base situation, Marshallian price elasticities can be evaluated numerically from scenarios where prices \(v_k\) differ from \(v_b\) one price at a time. And income elasticities can be estimated numerically from scenarios where all prices \(v_k\) are changed proportionally from \(v_b\). We consider the case where a priori information is available on the demand \(x_k\) in the \(k\)th scenario, a priori information which takes the form 2

\[
x_k \leq x_k \leq x_{k,M},
\]

\(x_k\) and \(x_{k,M}\) being respectively the lower and upper bound on \(x_k\) in the \(k\)th scenario, \(k \in K\). We assume that the bounds \(x_k\) and \(x_{k,M}\) are specified by the investigator in a way that is not too restrictive, so that they do not prevent the budget constraint \((v_k x_k \leq 1)\) from being binding at the optimum, \(k \in K\). 3

The prior bounds \(x_k\) and \(x_{k,M}\) can be incorporated into Eqs. (11) and (12). In the context of the inner-bound representation and using Eq. (11), consider the following problem:

\[
\begin{align*}
\text{Max} \quad & \sum_{k \in K} \left[ x_k \alpha_k + \beta_k + \xi_k (1 - v_k x_k) \right] : x_k \alpha_k + \beta_k \geq U_i, t \in T; \\
\text{Min} \quad & \sum_{k \in K} \left[ x_k \alpha_k + \beta_k + \xi_k (1 - v_k x_k) \right] : x_k \alpha_k + \beta_k \geq U_i, t \in T; \\
\alpha_k \geq 0; \xi_k \geq 0, x_k \leq x_{k,M} ; k \in K \\
\end{align*}
\]

\[
\begin{align*}
= \text{Max} \quad & \sum_{k \in K} \left[ x_k \alpha_k + \beta_k + \xi_k (1 - v_k x_k) + \eta_k (x_k - x_{k,L}) \right] : x_k \alpha_k + \beta_k \geq U_i, t \in T; \\
+ \rho_k (x_{k,M} - x_k) ; x_k \alpha_k + \beta_k \geq U_i, t \in T; \alpha_k \geq 0; \xi_k \geq 0; k \in K \\
= \text{Min} \quad & \sum_{k \in K} \left[ \beta_k + \xi_k - \eta_k x_k + \rho_k x_{k,M} ; x_k \alpha_k + \beta_k \geq U_i, t \in T; \alpha_k \geq 0; \xi_k \geq 0; k \in K \right].
\end{align*}
\]

(13)

where \(X = \{x_k, k \in K\}\), \(x_k\) is the \((n \times 1)\) vector of Lagrange multipliers for the

2 Note that such a priori information is local in nature since it is associated with a particular price/income scenario \(v_k\). Thus, it is up to the investigator to choose, through the choice of scenarios, the region of relative prices \(v\) where the a priori restrictions apply.

3 If the budget constraint \((v_k x_k \leq 1)\) were found to be non-binding at the optimum, the a priori restrictions \((x_{k,L} \leq x_k \leq x_{k,M})\) would then correspond to a rationing scheme.
constraints \((\alpha_k - \xi_k v_k + \eta_k - \rho_k \leq 0)\) in Eq. (13), \(\eta_k\) is the \((n \times 1)\) vector of Lagrange multipliers for the constraint \(x_{kL} \leq x_k\), and \(\rho_k\) is the \((n \times 1)\) vector of Lagrange multiplier for the constraint \(x_k \leq x_{kM}\). Eq. (13) is a reformulation of Eq. (11) with a priori bounds imposed on \(x_k\) across all scenarios \(k \in K\).

Similarly, in the context of the outer-bound representation and using Eq. (12), consider the following problem:

\[
\max_{x, U, \lambda} \left\{ \sum_{k \in K} \gamma_k \gamma_k \leq U_i + \lambda_i (v'_i x_k - 1), i \in T; \text{Eq. (6) }; v'_i x_k \leq 1; x_k \geq 0; \right. \\
\left. x_{kL} \leq x_k \leq x_{kM}; k \in K \right\},
\]  

(14)

where \(X = \{x_k; k \in K\}\). Eq. (14) is a reformulation of Eq. (12) with a priori bounds imposed on \(x_k\) across all scenarios \(k \in K\).

Eqs. (13) and (14) can be used to recover the unconditional inner and outer bounds on preferences under alternative scenarios after imposing a priori restrictions on the corresponding demands. Eq. (13) is a standard linear programming problem, while Eq. (14) is a nonlinear programming problem. They provide a convenient and empirically tractable method to incorporate a priori information in the nonparametric investigation of consumer preferences and demand behavior.

4. Welfare analysis

The two bounds \(W^i(x)\) and \(W^o(x)\) also provide a basis for conducting welfare analysis. This section focuses on the use of the nonparametric approach in evaluating cost-of-living and standard-of-living indexes for a consumer facing changing relative prices \(v = (p/I)\). For this purpose, define the expenditure function \(C^o(p, U_i)\) as follows:

\[
C^o(p, U_i) = \min_x \left[ p' x : W^o(x) \geq U_i, x \geq 0 \right],
\]  

(15)

where \(U_i\) is some reference utility level, and \(W^o\) is a particular representation of \(u(x)\) in Eq. (1). The solution of the optimization problem in Eq. (15) generates the Hicksian demand correspondences representing the quantities consumed under relative prices \(v\) and reference utility level \(U_i\). This solution also generates \(C^o(p, U_i)\), the smallest expenditure the consumer must pay in order to reach the reference utility level \(U_i\) under prices \(p\).

Consider a household facing two different economic conditions characterized respectively by the prices \(p_s\) and \(p_r\). Using the \(s\)th observation as a reference situation in welfare evaluation, the cost-of-living index \(P^o\) is:

\[
P^o(p_r, p_s, U_s) = C^o(p_r, U_s) / C^o(p_s, U_s) = C^o(p_r, U_s) / I_s.
\]  

(16a)

\(^4\) Note that Eq. (14) is only moderately nonlinear, the nonlinearities being associated with the constraints: \(\gamma_k \leq U_i + \lambda_i (v'_i x_k - 1), i \in T\).
Eq. (16a) is a Konius (1939) price index, defined as the ratio of expenditures holding utility constant under the two price situations. Alternatively, using the sth observation as a reference situation in welfare evaluation, the standard-of-living index $S^a$ is defined as:

$$S^a(p_1, U_i, U_s) = C^a(p_1, U_i)/C^a(p_1, U_s) = I_s/I_i.$$  \hspace{1cm} (16b)

Eq. (16b) compares two indifference curves by measuring the ratio of expenditures necessary to reach them at constant prices $p_i$. As such, Eq. (16b) can be interpreted as a money metric utility index. Note that the following relationship exists between the cost-of-living index (16a) and the standard-of-living index in Eq. (16b):

$$P^a(p_1, p_s, U_s)S^a(p_1, U_i, U_s) = [C^a(p_1, U_s)/C^a(p_1, U_s)][C^a(p_1, U_i)/C^a(p_1, U_i)] = C^a(p_1, U_i)/C^a(p_1, U_s) = I_s/I_i.$$ \hspace{1cm} (17)

Expression (17) shows that the cost-of-living index $P^a(p_1, p_s, U_s)$ in Eq. (16a) multiplied by the standard-of-living index $S^a(p_1, U_i, U_s)$ in Eq. (16b) equals the corresponding ratio of total expenditures $I_s/I_i$. This is consistent with the interpretation of $P^a$ as an aggregate price index, and $S^a$ as an aggregate quantity index. 5

The estimation of the cost-of-living index $P^a(p_1, p_s, U_s)$ in Eq. (16a) and of the standard-of-living index $S^a(p_1, U_i, U_s)$ in Eq. (16b) requires the evaluation of the expenditure function $C^a(.)$ in Eq. (15). The empirical issue then is the evaluation of this expenditure function. Given Eq. (8), we have two obvious candidates for $W^a(x)$ in Eq. (15): the unconditional inner-bound $W^i(x)$ in Eq. (7a), and the unconditional outer-bound $W^o(x)$ in Eq. (7b).

First, consider the inner-bound representation $W^i(x)$. Let $W^a(x) = W^i(x) = w^i(x, U^i)$ where $U^i$ is the inner-bound utility representation obtained in Eq. (13). Using Eq. (3) under non-satiation, the expenditure function in Eq. (15) takes the form (Afiart, 1987, p. 146): 6

$$C^i(p, U_i) = \min_x \left[ p^i x : w^i(x, U^i) \geq U_i ; x \geq 0 \right]$$

Note that Eqs. (16a) and (16b) could be alternatively defined as $P^a(p_1, p_s, U_s)$ and $S^a(p_1, U_i, U_s)$, respectively. Also, these indexes could be Malmquist indexes defined from the distance function (e.g., see Deaton, 1979). The extension of our analysis to the estimation of these alternative indexes is fairly straightforward.

As expected, $x_s$ is a solution to the minimization problem (18) evaluated at $p_s$. To see that, consider the following relationships:

$$C^i(p_s, U_s) = \min_x \left[ p^s x : w^i(x, U^i) \geq U_s ; x \geq 0 \right] \geq \min_x \left[ p^s x : U_i + \lambda_i (r_i x - 1) \geq U_s ; x \geq 0 \right] \text{from Eq. (A.2) in the Appendix,}$$

$$= p^s x_s \text{, under non-satiation.}$$

This implies that $x_s \in \arg \min_x \left[ p^s x : w^i(x, U^i) \geq U_s ; x \geq 0 \right]$. 

---

5 Note that Eqs. (16a) and (16b) could be alternatively defined as $P^a(p_1, p_s, U_s)$ and $S^a(p_1, U_i, U_s)$, respectively. Also, these indexes could be Malmquist indexes defined from the distance function (e.g., see Deaton, 1979). The extension of our analysis to the estimation of these alternative indexes is fairly straightforward.

6 As expected, $x_s$ is a solution to the minimization problem (18) evaluated at $p_s$. To see that, consider the following relationships:
\[
\begin{align*}
= \min_{x, \theta} \left[ p' x: \sum_{t \in T} U_t^i \theta_t \geq U_i; \sum_{t \in T} x \theta_t \leq x; \sum_{t \in T} \theta_t = 1; \theta_t \geq 0; t \in T; x \geq 0 \right] \\
= \min_{\theta} \left[ p' \left( \sum_{t \in T} x \theta_t \right): \sum_{t \in T} U_t^i \theta_t \geq U_i; \sum_{t \in T} \theta_t = 1; \theta_t \geq 0; t \in T; x \geq 0 \right],
\end{align*}
\]

(18)

where \( U_i = U_i^j \) obtained in Eq. (7a), Eq. (11) or Eq. (13). Eq. (18) constitutes a linear programming problem that provides a convenient basis for finding the expenditure function \( C^i(p, U_i) \) associated with \( W^i(x) \), the unconditional inner-bound representation of preferences, under the economic conditions given by the price vector \( p \). Solving problem (18) for specific values of \( p \) allows estimating Hicksian demand correspondences and the expenditure function \( C^i(p, U_i) \) associated with the inner bound \( W^i(x) \). The estimated expenditure function \( C^i(p, U_i) \) can in turn be used to evaluate the cost-of-living index \( P^i(p_i, p, U_i) \) from Eq. (16a), as well as the standard-of-living index \( S^i(p_i, U_i, U_j) \) from Eq. (16b).

Second, consider the outer-bound representation \( W^o(x) \). Let \( W^o(x) = w^o(x, U_i^o, \lambda^o) \), where \( U_i^o \) and \( \lambda^o \) are the outer-bound representation obtained in Eq. (14). Using Eqs. (7b) and (4') under non-satiation, the expenditure function in Eq. (15) takes the form (Afriat, 1987, p. 146) \(^7\):

\[
\begin{align*}
C^o(p, U_i) &= \min_x \left[ p' x: w^o(x, U_i^o, \lambda^o) \geq U_i; x \geq 0 \right] \\
&= \min_{x, \gamma} \left[ p' x: \gamma \geq U_i; \gamma \leq U_i^o + \lambda^o_i (v_i^o x - 1), t \in T; x \geq 0 \right] \\
&= \min_x \left[ p' x: U_i \leq U_i^o + \lambda^o_i (v_i^o x - 1), t \in T; x \geq 0 \right],
\end{align*}
\]

(19)

where \( U_i = U_i^o \) obtained in Eq. (7b), Eq. (11) or Eq. (13). Eq. (19) constitutes a linear programming problem that provides a convenient basis for finding the expenditure function \( C^o(p, U_i) \) associated with \( W^o(x) \), the unconditional outer-bound representation of preferences, under the economic conditions given by the price vector \( p \). Solving problem (19) for specific values of \( p \) allows estimating Hicksian demand correspondences and the expenditure function \( C^o(p, U_i) \) associated with the outer bound \( W^o(x) \). The estimated expenditure function \( C^o(p, U_i) \)

\(^7\) As expected, \( x_i \) is a solution of the minimization problem (19) evaluated at \( p_i \). To see that, consider the following relationships:

\[
\begin{align*}
C^o(p, U_i) &= \min_x \left[ p' x: w^o(x, U_i^o) \geq U_i; x \geq 0 \right] \\
&\geq \min_x \left[ p' x: U_i + \lambda^o_i (v_i^o x - 1) \geq U_i; x \geq 0 \right] \text{ from Eq. (4),}
\end{align*}
\]

This implies that \( x_i \in \text{argmin}_x \left[ p' x: w^o(x, U_i^o) \geq U_i; x \geq 0 \right] \).
can in turn be used to evaluate the cost-of-living index \( P^\alpha(p_s, p_s, U_s) \) from Eq. (16a), as well as the standard-of-living index \( S^\alpha(p_s, U_s, U_s) \) from Eq. (16b).

Following Afriat (1987, pp. 146–147), the expenditure functions \( C'(p, U_s) \) in Eq. (18) and \( C^\alpha(p, U_s) \) in Eq. (19) provide useful bounds, as stated next.

**Proposition 3.** Let \( \{U^i_s, s \in T\} \) be the solution of Eq. (13) and \( \{U^o_s, s \in T\} \) be the solution of Eq. (14). Then, the functions \( C'(p, U_s) \) in Eq. (18) and \( C^\alpha(p, U_s) \) in Eq. (19) provide, respectively, the outer-bound and inner-bound representation of the expenditure function:

\[
C^\alpha(p, U_s) \leq C(p, U_s) \leq C'(p, U_s), \tag{20}
\]

where \( C(p, U_s) \) is the expenditure function defined as \( C(p, U_s) \equiv \text{Min}_x \{p'x : W(x) \geq U_s, x \geq 0\} \), \( W(x) \) being a concave, non-satiated utility function rationalizing the data \( \{(x_t, u_t) : t \in T\} \) according to Eq. (1).

Proposition 3 follows directly from Eq. (15) and Proposition 2. Eq. (20) shows that \( C'(p, U_s) \) is the inner-bound estimate, while \( C^\alpha(p, U_s) \) is the outer-bound estimate of the expenditure function. From Eq. (16a), this implies that \( P^\alpha(p_s, p_s, U_s) \equiv C'(p_s, U_s) / I_s \) is an upper-bound estimate of the cost-of-living index, while \( P^\alpha(p_s, p_s, U_s) \equiv C^\alpha(p_s, U_s) / I_s \) is a lower-bound estimate of the cost-of-living index. And from Eq. (16b), this means that \( S^\alpha(p_s, U_s, U_s) \equiv I_s / C^\alpha(p_s, U_s) \) is a lower-bound estimate of the standard-of-living index, while \( S^\alpha(p_s, U_s, U_s) \equiv I_s / C^\alpha(p_s, U_s) \) is an upper-bound estimate of the standard-of-living index.

5. An application

In this section, we illustrate the empirical usefulness of the approach with an application to U.S. consumption data. The focus is on the use of nonparametric methods in the estimation of price and income elasticities of demand, and of cost-of-living and standard-of-living indexes using the inner-bound as well as the outer-bound representation of consumer preferences. We also illustrate how to incorporate prior information on consumer behavior (e.g., in terms of the sign of income elasticities) in nonparametric analysis. This is particularly useful when the gap between the inner-bound and the outer-bound representations of preferences is found to be large.

The analysis relies on annual aggregate U.S. personal consumption expenditures per capita and consumer price indexes for all urban consumers (CPI, 1982–84 base) for 1953–1992. Eight aggregate consumption categories are considered: (1) durables; (2) food; (3) clothing; (4) fuel and utilities; (5) other
nondurables; (6) medical services; (7) transportation services; and (8) other services. The 1950–92 expenditure data are from the U.S. Department of Commerce. The 1950–92 CPI data are from the U.S. Department of Labor, Bureau of Labor Statistics. Implicit per capita quantity indexes for each consumption good are obtained by dividing the per-capita expenditures by their associated price index. Thus, by definition, the product of price index and quantity index equals the corresponding expenditure.

First, the Afriat inequalities (2a) and (2b) are found to be feasible, using the aggregate per capita U.S. consumption data. This implies the existence of a utility function that can rationalize the data. The exact nature of consumer preferences and their behavioral implications are then investigated using Eqs. (13) and (14). These optimization problems are solved numerically using GAMS-MINOS software. The year 1972 is used as the base observation \((x_b, v_b)\). The a priori bounds \(x_{k_L}\) and \(x_{k_M}\) in Eqs. (13) and (14) are specified as follows: (1) the predicted demands \(x_k\) are restricted to lie within the range of the data; \(^9\) and (2) all goods are assumed to exhibit non-negative income effects. \(^{10}\) The numerical elasticities are estimated using the percentage response to 20 percent price changes from the base price \(v_b\) for each commodity. Income elasticities are obtained using the percentage response to 20 percent changes in all prices (i.e., 20 percent income changes). \(^{11}\) All elasticities reported below are estimated as the average of two elasticities: one obtained from a 20 percent increase, the other from a 20 percent decrease in the relevant variable. As a result, 18 scenarios are evaluated in Eqs. (13) and (14): 8 for price increases, 8 for price decreases, 1 for income increase, and 1 for income decrease.

Table 1 summarizes the Marshallian price and income elasticities associated with the inner-bound equation (Eq. (13)) as well as outer-bound equation (Eq. (14)) representation of preferences. Recall that these elasticities are globally consistent with the preference maximization hypothesis by construction. In gen-

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\(^8\) The following parameters are used: \(\epsilon = 0.1\) in Eq. (6a), and \(K = 1000\) in Eq. (6d).

\(^9\) This avoids corner solutions. Indeed, without a priori restrictions, our piece-wise linear representations of preferences would imply that getting outside the range of the data is typically associated with zero consumption for at least one commodity. At the aggregate, predicting zero consumption for any commodity does not appear realistic. On this basis, restricting predicting demands \(x_k\) to fall within the range of the data appears reasonable.

\(^{10}\) Estimating price and income elasticities without imposing these a priori restrictions generated estimates that violated such restrictions. For example, in an unrestricted model, some commodities were found to be ‘inferior’ goods, with negative income elasticities. This seemed rather unrealistic, and motivated us to develop the restricted estimation method proposed in Eqs. (13) and (14).

\(^{11}\) 20 percent changes are chosen for two reasons. First, they cover a significant portion of the data range. Second, they seem large enough to generate meaningful changes in behavior (given the piece-wise linear representations of consumer preferences (7a) and (7b)).
Table 1
Marshallian price and income elasticities associated with the inner and outer bound representation of preferences: \(^a\)

<table>
<thead>
<tr>
<th>Prices:</th>
<th>Durables</th>
<th>Food</th>
<th>Clothing</th>
<th>Fuel-util</th>
<th>O-nondur</th>
<th>Med. serv</th>
<th>Transp</th>
<th>O-serv</th>
<th>Income</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantities:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Durables</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.358</td>
<td>−0.341</td>
<td>0.009</td>
<td>−0.070</td>
<td>−0.167</td>
<td>0.038</td>
<td>−0.066</td>
<td>−0.472</td>
<td>1.562</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>−2.486</td>
<td>−1.214</td>
<td>1.043</td>
<td>1.057</td>
<td>1.551</td>
<td>−0.244</td>
<td>0.263</td>
<td>0.869</td>
<td>1.055</td>
</tr>
<tr>
<td><strong>Food</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>0.081</td>
<td>−0.042</td>
<td>0.068</td>
<td>0.001</td>
<td>0.030</td>
<td>0.084</td>
<td>−0.007</td>
<td>−0.020</td>
<td>0.252</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>−0.311</td>
<td>−0.549</td>
<td>0.340</td>
<td>0.120</td>
<td>0.285</td>
<td>−0.086</td>
<td>−0.127</td>
<td>0.000</td>
<td>0.067</td>
</tr>
<tr>
<td><strong>Clothing</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.228</td>
<td>−0.255</td>
<td>−0.438</td>
<td>−0.059</td>
<td>−0.112</td>
<td>−0.531</td>
<td>−0.049</td>
<td>−0.552</td>
<td>1.608</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>2.269</td>
<td>0.345</td>
<td>−3.110</td>
<td>−1.164</td>
<td>−2.409</td>
<td>−0.304</td>
<td>0.076</td>
<td>0.000</td>
<td>0.653</td>
</tr>
<tr>
<td><strong>Fuel-utilities</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.214</td>
<td>−0.172</td>
<td>−0.244</td>
<td>−0.063</td>
<td>−0.107</td>
<td>−0.285</td>
<td>−0.021</td>
<td>−0.687</td>
<td>0.223</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>−0.634</td>
<td>0.476</td>
<td>−0.753</td>
<td>−1.699</td>
<td>0.501</td>
<td>0.884</td>
<td>−0.909</td>
<td>−0.142</td>
<td>0.938</td>
</tr>
<tr>
<td><strong>Other nondurables</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.176</td>
<td>−0.086</td>
<td>0.284</td>
<td>0.009</td>
<td>−0.102</td>
<td>0.353</td>
<td>−0.015</td>
<td>0.049</td>
<td>0.802</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>1.139</td>
<td>1.350</td>
<td>−0.529</td>
<td>−0.166</td>
<td>−1.675</td>
<td>0.240</td>
<td>1.017</td>
<td>−0.041</td>
<td>0.688</td>
</tr>
<tr>
<td><strong>Medical services</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.254</td>
<td>−0.444</td>
<td>−0.616</td>
<td>−0.084</td>
<td>−0.163</td>
<td>−0.785</td>
<td>−0.082</td>
<td>−0.735</td>
<td>1.951</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>−1.151</td>
<td>−2.489</td>
<td>−0.172</td>
<td>0.826</td>
<td>−0.972</td>
<td>−2.632</td>
<td>0.493</td>
<td>2.141</td>
<td>2.632</td>
</tr>
<tr>
<td><strong>Transportation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.140</td>
<td>−0.381</td>
<td>0.129</td>
<td>−0.014</td>
<td>−0.117</td>
<td>0.113</td>
<td>−0.070</td>
<td>−0.272</td>
<td>1.389</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>2.060</td>
<td>−0.880</td>
<td>−0.615</td>
<td>−0.855</td>
<td>0.000</td>
<td>−0.068</td>
<td>−2.060</td>
<td>2.060</td>
<td>0.889</td>
</tr>
<tr>
<td><strong>Other services</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Inner rep.</td>
<td>−0.135</td>
<td>−0.185</td>
<td>−0.086</td>
<td>−0.046</td>
<td>−0.068</td>
<td>−0.098</td>
<td>−0.037</td>
<td>−0.407</td>
<td>1.066</td>
</tr>
<tr>
<td>Outer rep.</td>
<td>0.193</td>
<td>0.561</td>
<td>0.050</td>
<td>−0.262</td>
<td>0.041</td>
<td>0.474</td>
<td>−0.179</td>
<td>1.635</td>
<td>1.440</td>
</tr>
</tbody>
</table>

\(^a\) The first row reports the elasticity estimates associated with the inner-bound representation of preferences. The elasticity estimates associated with the outer-bound representation of preferences are presented on the second row. All elasticities are evaluated at the 1972 data point.

...eral, these results indicate that the elasticities vary widely with the bounds. The demand for food is consistently found to be price inelastic, with an own-price elasticity of −0.042 (inner-bound representation) and −0.549 (outer-bound representation). The inner-bound representation always gives relatively small own-price elasticities: the Marshallian own-price elasticities are all inelastic, ranging from −0.042 for food to −0.785 for medical services. In contrast, many Marshallian own-price elasticities associated with the outer bound are elastic: they range from ...
Table 2
Cost-of-living and standard-of-living indexes, base year = 1972

<table>
<thead>
<tr>
<th>Year</th>
<th>Cost-of-living index</th>
<th>Standard-of-living index</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p^i$</td>
<td>$p_o^i$</td>
</tr>
<tr>
<td></td>
<td>(Upper-bound estimate)</td>
<td>(Lower-bound estimate)</td>
</tr>
<tr>
<td>1953</td>
<td>0.609</td>
<td>0.587</td>
</tr>
<tr>
<td>1954</td>
<td>0.615</td>
<td>0.597</td>
</tr>
<tr>
<td>1955</td>
<td>0.617</td>
<td>0.603</td>
</tr>
<tr>
<td>1956</td>
<td>0.628</td>
<td>0.615</td>
</tr>
<tr>
<td>1957</td>
<td>0.653</td>
<td>0.642</td>
</tr>
<tr>
<td>1958</td>
<td>0.673</td>
<td>0.663</td>
</tr>
<tr>
<td>1959</td>
<td>0.682</td>
<td>0.674</td>
</tr>
<tr>
<td>1960</td>
<td>0.697</td>
<td>0.692</td>
</tr>
<tr>
<td>1961</td>
<td>0.703</td>
<td>0.698</td>
</tr>
<tr>
<td>1962</td>
<td>0.714</td>
<td>0.709</td>
</tr>
<tr>
<td>1963</td>
<td>0.725</td>
<td>0.721</td>
</tr>
<tr>
<td>1964</td>
<td>0.734</td>
<td>0.730</td>
</tr>
<tr>
<td>1965</td>
<td>0.747</td>
<td>0.743</td>
</tr>
<tr>
<td>1966</td>
<td>0.771</td>
<td>0.767</td>
</tr>
<tr>
<td>1967</td>
<td>0.792</td>
<td>0.790</td>
</tr>
<tr>
<td>1968</td>
<td>0.826</td>
<td>0.825</td>
</tr>
<tr>
<td>1969</td>
<td>0.871</td>
<td>0.870</td>
</tr>
<tr>
<td>1970</td>
<td>0.928</td>
<td>0.926</td>
</tr>
<tr>
<td>1971</td>
<td>0.969</td>
<td>0.968</td>
</tr>
<tr>
<td>1972</td>
<td>1.000</td>
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- 0.549 for food to - 3.110 for clothing. Intuitively, this reflects the fact that the outer-bound representation of preferences tends to be 'less concave' (and thus more price responsive) than the inner-bound representation. From Proposition 2, these large differences between own-price elasticities according to the utility bound suggest that there exists a whole family of preference functions that can rationalize the data. It clearly indicates that there is more than one demand correspondence (and its associated elasticities) that is consistent with both the data and consumption theory.

The income elasticities associated with the nonparametric bounds are reported in Table 1. Income elasticities also vary with the bounds. The income elasticity of food is found to be 0.252 (inner-bound representation) and 0.067 (outer-bound representation). The income elasticities associated with the inner-bound representation of preferences are sometimes larger (e.g., clothing), sometimes smaller (e.g., fuel and utilities) than the corresponding elasticities associated with the outer-bound representation (see Table 1). Income elasticities range from 0.223 (fuel and utilities) to 1.951 (medical services) using the inner-bound representation, and from 0.067 (food) to 2.632 (medical services) using the outer-bound representation. Again, these results indicate that there exists a fairly wide range of preferences that are consistent with both the theory and the data. The cross-price elasticities reported in Table 1 indicate similar conclusions.

Next, we investigate the implications of our results for welfare analysis following the approach discussed in Section 4. Using the utility levels $U$ obtained from Eqs. (13) and (14), the expenditure functions (18) and (19) are evaluated, using 1972 as the base year ($s = 1972$). Note that, since the $U$'s incorporate the a priori restrictions imposed in Eqs. (13) and (14), it follows that the associated preference structure also reflects this a priori information. The cost-of-living and standard-of-living indexes given in Eqs. (16a) and (16b) are reported in Table 2 for both the inner-bound and the outer-bound representation of preferences. As shown in Section 4, these estimates give lower-bound and upper-bound measures of the corresponding indexes. Table 2 indicates that the gap between the two bounds is in general fairly small. This is consistent with the cost-of-living results obtained by Varian (1982). Table 2 indicates a fairly steady increase in the U.S. standard of living between 1953 and 1992. These results illustrate the general usefulness of the nonparametric approach in demand analysis.

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12 All estimates of income elasticities are obviously non-negative since such restrictions are imposed in Eqs. (13) and (14).

13 In the interpretation of Table 1, one should be careful to distinguish between bounds on preferences and bounds on demand elasticities. For example, the income elasticities reported in Table 1 are income elasticities associated with the inner-bound and outer-bound representations of preferences. They provide useful information on the range of income elasticities that are consistent with both the data and the theory. However, in general, they are not the widest possible bounds on these elasticities.
6. Concluding remarks

This paper investigates the use of nonparametric methods in the analysis of consumer behavior. The nonparametric approach allows for a general representation of preferences that does not require ad hoc specification of functional form for utility or demand functions. We show how nonparametric bounds on preferences can be computed, along with their corresponding price and income elasticities. Also, we show how to incorporate a priori information on demand behavior (e.g., non-negative income effects) using the nonparametric approach. The approach is illustrated in a nonparametric estimation of U.S. preferences for eight commodities. This generates nonparametric representations of consumer preferences and demand response that are globally consistent with consumer theory (i.e., the theory holds at every data point). The methodology is easy to apply and can handle a large number of commodities, hence reducing the need to impose a priori aggregation structure. The estimated price and income elasticities of demand appear reasonable. These elasticities are found to vary greatly between the inner-bound and outer-bound representation of preferences. This clearly indicates that there is more than one set of demand correspondences that is consistent with both the data and the theory. Cost-of-living and standard-of-living indexes are also evaluated. The estimated bounds on these indexes are found to be fairly tight. These findings suggest that nonparametric methods can be quite useful for applied consumption and welfare analysis.

While our analysis illustrates well the potential value of the nonparametric approach, its limitations should be kept in mind. One of its shortcomings is the fact it is not statistically-based. As a result, the nonparametric method discussed in this paper lacks the ability to conduct statistical testing. However, the evaluation of nonparametric bounds can provide useful information of the range of preferences and demand response that is consistent with both the data and the theory. Given its generality, its flexibility, and its relative ease of application, the proposed method has much to offer. We hope that it will help stimulate additional research evaluating the empirical usefulness of nonparametric methods.

Acknowledgements

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Appendix A

The proofs of Lemma 1, Lemma 2, and Proposition 1 can be found in Afriat (1987). They are presented here for the sake of completeness.
Proof of Lemma 1. First, consider Eq. (3). When evaluated at \( x = x_s \), note that \( \theta_s = 1 \) (= 0) for \( s = t \) \(( s \neq t )\) is a feasible solution to the maximization problem (3). It follows that
\[
\omega^i( x_s, U ) \geq U_s, \text{ for all } s \in T. \tag{A.1}
\]

Next, turn to Eq. (3'). Consider choosing \( \beta = [ U_s - \lambda_s v'_s x_s ] \) and \( \alpha_s = \lambda_s v'_s \) in Eq. (3'). The first set of constraints in Eq. (3') then becomes: \( \lambda_s v'_s x_s + U_s - \lambda_s v'_s x_s \geq U_s \), which, under non-satiation, is Eq. (2b) with \( s \) and \( t \) interchanged. Thus, under Eq. (2b), this choice of \( \alpha_s \) and \( \beta \) is always feasible. Also, note that, under non-satiation, this feasible choice to the minimization problem (3') yields the value of the objective function: \( U_s + \lambda_s [ v'_s x - 1 ] \). It follows that
\[
\omega^i( x, U ) \leq U_s + \lambda_s [ v'_s x - 1 ], \text{ for all } s \in T. \tag{A.2}
\]

When \( x = x_s \), it follows from Eq. (A.2) that, under non-satiation,
\[
\omega^i( x_s, U ) \leq U_s, \text{ for all } s \in T. \tag{A.3}
\]

From Eqs. (A.1) and (A.3), \( \omega^i( x_s, U ) \) is both an upper bound and a lower bound to \( U_s \), implying that \( \omega^i( x_s, U ) = U_s \) for all \( s \in T \). To show that \( \omega^i( x, U ) \) is in fact a utility function which rationalizes the data, consider the following relationships:
\[
\begin{align*}
\max_x \{ \omega^i( x, U ) : v'_s x \leq 1, x \geq 0 \} & \leq \max_x \{ U_s + \lambda_s ( v'_s x - 1 ) : v'_s x \leq 1, x \geq 0 \} \\
& \text{from Eq. (A.2),} \\
& = U_s \text{ when } x = x_s \text{ under non-satiation.}
\end{align*}
\]

This shows that \( x_s \) is a solution to the utility maximization problem, \( x_s \in \arg \max \{ \omega^i( x, U ) : v'_s x \leq 1, x \geq 0 \} \), with \( \omega^i( x, U ) \) as a representation of consumer preferences satisfying \( \omega^i( x_s, U ) = U_s, s \in T \).

Proof of Lemma 2. First, consider Eq. (4). When evaluated at \( x = x_s \), it is clear that, under non-satiation,
\[
\omega^o( x_s, U, \lambda ) \leq U_s. \tag{A.4}
\]

Next, consider Eq. (4'). It is clear from Eq. (2b) that, when \( x = x_s \), choosing \( \gamma = U_s \) in Eq. (4') is feasible. It follows from the maximization problem (4') that
\[
\omega^o( x_s, U, \lambda ) \geq U_s. \tag{A.5}
\]

From Eqs. (A.4) and (A.5), \( \omega^o( x_s, U, \lambda ) \) is both an upper bound and a lower bound to \( U_s \), implying that \( \omega^o( x_s, U, \lambda ) = U_s \) for all \( s \in T \). To show that \( \omega^o( x, U, \lambda ) \) is in fact a utility function which rationalizes the data, consider the following relationships:
\[
\begin{align*}
\max_x \{ \omega^o( x, U, \lambda ) : v'_s x \leq 1, x \geq 0 \} & \leq \max_x \{ U_s + \lambda_s ( v'_s x - 1 ) : v'_s x \leq 1, x \geq 0 \} \\
& \text{from Eq. (4),} \\
& = U_s \text{ when } x = x_s \text{ under non-satiation.}
\end{align*}
\]
This shows that $x_s$ is a solution to the utility maximization problem, $x_s \in \arg\max_x \{\mathcal{w}(x, U, \lambda); v'_s x \leq 1, x \geq 0\}$, with $\mathcal{w}(x, U, \lambda)$ as a representation of consumer preferences satisfying $\mathcal{w}(x_s, U, \lambda) = U_s$, $s \in T$.

**Proof of Proposition 1.** By definition, any non-decreasing and concave function $w(x)$ necessarily satisfies the conditions:

\[
\begin{align*}
\text{(non-decreasing)} & \quad \sum_{t \in T} x_t \theta_t \leq x \text{ implies that } w\left(\sum_{t \in T} x_t \theta_t\right) \leq w(x), \\
\text{(concave)} & \quad \sum_{t \in T} w(x_t) \theta_t \leq w\left(\sum_{t \in T} x_t \theta_t\right),
\end{align*}
\]

where $\sum_{t \in T} \theta_t = 1$, $\theta_t \geq 0$, $t \in T$. Combining these two results gives:

\[
\sum_{t \in T} x_t \theta_t \leq x, \quad \sum_{t \in T} \theta_t = 1, \theta_t \geq 0 \text{ implies that } \sum_{t \in T} w(x_t) \theta_t \leq w(x). \tag{A.6}
\]

If $w(x_t) = U_t$ for all $t$, it follows from Eq. (A.6) and the maximization problem (3) that

\[
w(x) \geq w'(x, U) \text{ for all } x,
\]

implying that $w'(x, U)$ is the tightest inner bound representation of $u(x)$ for a given set of $U$ and $\lambda$.

And from the maximization problem (4'), it clear that $\mathcal{w}(x, U, \lambda)$ provides the tightest upper bound representation of utility levels consistent with Afriat inequalities (2a) and (2b).

**References**


