CHAPTER IV: DUALITY IN LINEAR PROGRAMMING

Economic theory indicates that scarce (limited) resources have value. For example, prime agricultural van capacity is limited and has value (a rental price). On the other hand, air is effectively unlimited and therefore does not have a market value. In LP models, limited resources are allocated, so they should be, valued. Whenever we solve an LP problem, we implicitly solve two problems: the primal resource allocation problem, and the dual resource valuation problem. This chapter covers the resource valuation, or as it is commonly called, the Dual LP problem and its relationship to the original, primal, problem.

4.1 Basic Duality

The study of duality is very important in LP. Knowledge of duality allows one to develop increased insight into LP solution interpretation. Also, when solving the dual of any problem, one simultaneously solves the primal. Thus, duality is an alternative way of solving LP problems. However, given today's computer capabilities, this is an infrequently used aspect of duality. Therefore, we concentrate on the study of duality as a means of gaining insight into the LP solution. We will also discuss the ways that primal decision variables place constraints upon the resource valuation information. The Primal problem can be written as:

$$
\begin{array}{llccl}
Max & \sum_j c_j X_j & & & \\
s.t. & \sum_j a_{ij} X_j & \leq & b_i & \text{for all } i \\
 & X_j & \geq & 0 & \text{for all } j
\end{array}
$$

Associated with this primal problem is a dual resource valuation problem. The dual of the above problem is

$$
\begin{array}{llll}
\text{Min} & \sum_i U_i b_i & & \\
\text{s.t.} & \sum_i U_i a_{ij} & \geq & c_j \quad \text{for all j} \\
& U_i & \geq & 0 \quad \text{for all i}
\end{array}
$$

where U_i are the dual variables.

If the primal problem has n variables and m resource constraints, the dual problem will have m variables and n constraints. There is a one-to-one correspondence between the primal constraints and the dual variables; i.e., U_1 is associated with the first primal constraint, U_2 with the second primal constraint, etc. As we demonstrate later, dual variables (U_i) can be interpreted as the marginal value of each constraint's resources. These dual variables are usually called shadow prices and indicate the imputed value of each resource. A one-to-one correspondence also exists between the primal variables and the dual constraints; X_1 is associated with the first dual constraint ($\sum_i U_i a_{i1} \geq c_1$), X_2 is associated with the second dual constraint ($\sum_i U_i a_{i2} \geq c_2$), etc.

An example aids in explaining the dual. Consider the primal model:

The associated dual problem is

$$
\begin{array}{lccccccc}
 & (van\,cap) & & (labor) & & & & (profits) \\
\min & 12U_1 & + & 280U_2 & & & & (\text{Re}\,source\,Payments) \\
s.t. & U_1 & + & 25U_2 & \geq & 2000 & & (X_{fancy}) \\
 & U_1 & + & 20U_2 & \geq & 1700 & & (X_{fine}) \\
 & U_1 & + & 19U_2 & \geq & 1200 & & (X_{new}) \\
 & U_1 & , & U_2 & \geq & & & (nonegativity)
\end{array}
$$

The dual problem economic interpretation is important. The variable U_1 gives the marginal value of the first resource, or van capacity. Variable U_2 gives the marginal value of the second resource, or labor in this case. The first dual constraint restricts the value of the resources used in producing a unit of X_{fancy} to be greater than or equal to the marginal revenue contribution of X_{fancy}. In the primal problem X_{fancy} uses one unit of van capacity and 25 units of labor, returning \$2000, while the dual problem requires van capacity use times its marginal value ($1U_1$) plus labor use times its marginal value ($25U_2$) to be greater than or equal to the profit earned when one unit of X_{fancy} is produced (2000). Similarly, constraint 2 requires the marginal value of van capavity plus 20 times the marginal value of labor to be greater than or equal to 1700, which is the amount of profit earned by producing X_{fine} and the third constaint does the same for X_{new}. Thus, the dual variable values are constrained such that the marginal value of the resources used by each primal variable is no less than the marginal profit contribution of that variable.

Now suppose we examine the objective function. This function minimizes the total marginal value of the resources on hand. In the example, this amounts to the van capacity endowment times the marginal value of van capacity ($12U_1$) plus the labor endowment times the marginal value of labor ($320U_2$).

Thus, the dual variables arise from a problem minimizing the marginal value of the resource endowment subject to constraints requiring that the marginal value of the resources used in producing each product must be at least as great as the marginal value of the product. This can be viewed as the problem of a resource purchaser in a perfectly competitive market. Under such circumstances, the purchaser would have to pay at least as much for the resources as the value of the goods produced using those resources. However, the purchaser would try to minimize the total cost of the resources acquired.

The resultant dual variable values are measures of the marginal value of the resources. The objective function is the minimum value of the resource endowment. Any slack in the constraints is the amount that cost exceeds revenue.

4.2 Primal-Dual Solution Inter-Relationships

Several relationships exist between primal and dual solutions which are fundamental to understanding duality and interpreting LP solutions.

$$\begin{array}{llll} \textit{Primal} & & & \\ Max & CX & & \\ s.t. & AX & \leq & b \\ & X & \geq & 0 \end{array} \qquad \begin{array}{llll} \textit{Dual} & & & \\ Min & U'b & & \\ s.t. & U'A & \geq & C \\ & U & \geq & 0 \end{array}$$

First, let us introduce some notation. The primal dual pair of LP problems in matrix form is. Now let us examine how the problems are related.

4.2.1 Objective Function Interrelationships

Suppose we have any two feasible primal and dual solutions X^*, U^* and we want to determine the relationship between the objective functions CX^* and $U^{*\prime}b$. We know the feasible solutions must satisfy

$$\begin{array}{lll} AX^* & \leq & b \\ X^* & \geq & 0 \end{array} \quad \text{and} \quad \begin{array}{lll} U^{*\prime}A & \geq & C \\ U^* & \geq & 0 \end{array}$$

To determine the relationship, we take the above constraint inequalities (not the non-negativity conditions) and premultiply the left one by $U^{*\prime}$ while postmultiplying the right one by X^*.

$$U^{*\prime}AX^* \geq CX^*$$

$$U^{*\prime}AX^* \leq U^{*\prime}b$$

Noting that the term $U^{*\prime}AX^*$ is common to both inequalities we get

$$CX^* \leq U^{*\prime}AX^* \leq U^{*\prime}b$$

$$CX^* \leq U^{*\prime}b$$

This shows that the dual objective function value is always greater than or equal to the primal objective function value for any pair of feasible solutions.

4.2.2 Constructing Dual Solutions

We can construct an optimal dual solution from an optimal primal solution. Suppose an optimal primal solution is given by $X_B^* = B^{-1}b$ and $X_{NB}^* = 0$. This solution must be feasible; i.e., $X_B^* = B^{-1}b \geq 0$. It also must have $X_{NB} \geq 0$ and must satisfy nonnegative reduced cost for the nonbasic variables $C_B B^{-1} A_{NB} - C_{NB} \geq 0$. Given this, suppose we try $U^{*'} = C_B B^{-1}$ as a potential dual solution.

First, let us investigate whether this is a feasible solution in the dual constraints. To be feasible, we must have $U^{*'} A \geq C$ and $U^* \geq 0$. If we set $U^{*'} = C_B B^{-1}$, then we know $U^{*'} A_{NB} - C_{NB} \geq 0$ because at optimality this is exactly equivalent to the reduced cost criteria, i.e., $C_B B^{-1} A_{NB} - C_{NB} \geq 0$. Further, we know for the basic variables the reduced cost is $C_B B^{-1} A_B - C_B = C_B B^{-1} B - C_B = C_B - C_B = 0$, so $U^{*'} B = C_B$. By unifying these two statements, we know when $U^* = C_B B^{-1}$ then the dual inequalities $U^{*'} A \geq C$ are satisfied.

Now we need to know if the dual nonnegativity conditions $U \geq 0$ are satisfied. We can look at this by looking at the slacks in the problem. For the slacks, A contains an identity matrix and the associated entries in C are all 0's. Thus, for the part of the $UA \geq C$ that covers the slacks and since we know that $U^{*'} A_S \geq C_S$ where A_S and C_S are the portions of A and C relevant to the slacks. Substituting in the known structure of A_S and C_S, i.e., $A_S = I$ and $C_S = 0$ yields $U^{*'} \geq 0$ or $U \geq 0$. So the U's are non-negative. Thus, $U^{*'} = C_B B^1$ is a feasible dual solution.

Now the question becomes, is this choice optimal? In this case the primal objective function Z equals $CX^* = C_B X_B^* + C_{NB} X_{NB}^*$ and since X_{NB}^* equals zero, then $Z = C_B B^{-1} b + C_{NB} 0 = C_B B^{-1} b$. Simultaneously, the dual objective equals $U^{*'} b = C_B B^{-1} b$ which equals the primal objective. Therefore, the primal and dual objectives are equal at optimality. Furthermore, since the primal objective must be less than or equal to the dual objective for any feasible pair of solutions and since they are equal, then clearly CX^* cannot get any larger nor can $U^{*'} b$ get any smaller, so they must both be optimal. Therefore, $C_B B^{-1}$ is an optimal dual solution. This demonstration shows that given the solution from the primal the dual solution can simply be computed without need to solve the dual problem.

In addition given the derivation in the last chapter we can establish the interpretation of the dual variables. In particular, since the optimal dual variables equal $C_B B^{-1}$ (which are called the primal shadow prices) then the dual variables are interpretable as the marginal value product of the resources since we showed

$$\frac{\partial Z}{\partial b} = C_B B^1 = U^{*'}.$$

4.2.3 Complementary Slackness

Yet another, interrelationship between the primal and dual solutions is the so called complementary slackness relation. This states that given optimal values U^* and X^* then

$$U^{*\prime}(b - AX^*) \quad = \quad 0$$

$$(U^{*\prime}A - C)\,X^* \quad = \quad 0.$$

This result coupled with the primal and dual feasibility restrictions ($U \geq 0$; $UA \geq C$; $AX \leq b$; $X \geq 0$) implies that (in the absence of degeneracy and multiple optimal solutions) for each constraint, either the resource is fully used (b_i - $(AX)_i = 0$) with the associated dual variable (U_i^*) nonzero, or the dual variable is zero with associated unused resources (b_i - (AX_j)) being nonzero. Alternatively, for each variable (again ignoring degeneracy) at optimality, either the variable level (X_j) is non-zero with zero reduced cost ($U^{*\prime}$ $A)_j$ - $c_j = 0$) or the variable is set to zero with a non-zero reduced cost.

This result may be proven using matrix algebra. Given optimal primal (X^*) and dual (U^*) solutions

$$\text{Let} \quad \alpha \quad = \quad U^{*\prime}(b - AX^*)$$

$$\beta \quad = \quad (U^{*\prime}A - C)\,X^*.$$

Now suppose we add together □ + □□and examine the result. Name □ + □, equals

$$U^{*\prime}\,(b - AX^*) + (\,U^{*\prime}\,A - C)X^*$$

which equals

$$U^{*\prime}\,b - U^{*\prime}\,AX^* + U^{*\prime}\,AX^* - CX^* = U^{*\prime}\,b - CX^*.$$

We know this equals zero at optimality. Further, we know that both □ and □ will be nonnegative, since $AX^* \leq b$, $U^* \geq 0$, $U^{*\prime}$ $A - C \geq 0$ and $X^* \geq 0$, thus □ + □ can be equal to zero if and only if □ and □ are both equal to zero. Thus, complementary slackness must hold at optimality.

The complementary slackness conditions are interpretable economically. The $U^{*\prime}$ $(AX^*$ - $b) = 0$ condition implies that the resource will: a) be valued at zero if it is not fully utilized (i.e., $AX^* \leq b$ □ $U^* = 0$) or b) have a nonzero value only if it is fully utilized (i.e., $AX = b$ □ $U > 0$ [note a zero value could occur]). Thus, resources only take on value only when they have been exhausted. The condition $(U'A \;\; C)X = 0$ implies that a good will only be produced if its reduced cost is zero (i.e., $X > 0$ □ $A - C = 0$) and that only zero X's can have a reduced cost (i.e., U' $A - C > 0$ □ $X = 0$). This last result also shows the returns (C) to every nonzero variable are fully allocated to the shadow prices (U) times the amount of resources (A) used in producing that activity (i.e., U' $A=C$).

4.2.4 Zero Profits

We have noted that U_i is the imputed marginal value of resource i and b_i is its endowment. Thus,

$U_{i'}$ b_i is sometimes called the "payment" to resource i. When we sum over all m resources, the dual objective function can be interpreted as the total imputed value of the resource endowment. If the total imputed value of the resources is viewed as a "cost", then it makes sense that firm should seek to find $U_1, U_2, \ldots, U_m$, which minimizes $\sum U_i b_i$. However, at optimality the dual objective equals that of the primal

$$\sum_i U_i b_i = \sum_j c_j X_j .$$

Thus, total payments to the resources will equal the profit generated in the primal problem. All profits are allocated to resource values, and the solution insures that the imputed rate of the resources allocated by the primal problem are such that their total value equals total profits. Thus, if the firm had to pay U for the resources, zero profits would result.

4.2.5 Finding the Dual Solution Information

When you have solved the dual, you have solved the primal. Thus given the optimal B^{-1}, the optimal dual variables are the primal shadow prices $C_B B^{-1}$ without any need for solution. In general, one can show that the following correspondence holds (see Hadley (1962) or Bazaraa et al.).

Primal Solution Item	Dual Solution Item
Objective function	Objective function
Shadow prices	Variable values
Slacks	Reduced costs
Variable values	Shadow prices
Reduced costs	Slacks

For example, if one wants to know the optimal values of the dual slacks, those values are the primal reduced costs.

4.3 Duality Under Other Model Forms

In the preceding discussion, the primal problem has always taken on standard form. We have seen that given a LP problem

$$\begin{aligned} \text{Max} \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & \sum_j a_{ij} x_j \leq b_i, \quad \text{for all } i \\ & x_j \geq 0, \quad \text{for all } j \end{aligned}$$

that its dual will always be

$$\begin{aligned} \text{Min} \quad & \sum_i u_i b_i \\ \text{s.t.} \quad & \sum_i u_i a_{ij} \ \geq \ c_j, \ \text{for all i} \\ & \quad u_i \ \geq \ 0, \quad \text{for all j.} \end{aligned}$$

Note that if the primal problem contains n variables and m constraints, the dual problem has m variables and n constraints. But not all problems have less than or equal to constraints and The dual for problems which are not in standard form can be written in two ways. One may convert a problem in non-standard form to reformulate it into standard form then write the dual, or one can learn primal-dual relationships for alternative primal problem forms. We discuss the second approach first.

The form of the primal constraints determines the restrictions on the sign of the associated dual variable. If the primal objective is to maximize, each $\leq$ constraint has a corresponding non-negative (≥ 0) dual variable. Each $\geq$ constraint has a corresponding non-positive (≤ 0) dual variable. Why? If a ($\geq$) constraint is binding in a maximize primal, it follows that reducing the RHS of the constraint would make the constraint less binding and could only improve or leave unaffected the optimal objective function value. Thus, the objective function value is unchanged or decreases if the RHS of the constraint is increased and the associated dual variable is non-positive. An equality constraint in a primal problem gives a dual variable which is unrestricted in sign. The optimal solution to the primal problem must lie on the equality constraint. An outward shift in the constraint could either increase or decrease the objective function, thus the corresponding dual variable is unrestricted in sign. These relationships are summarized in the first part of Table 4.1.

In regards to the primal variables, if the primal objective is to maximize, then each non-negative primal variable gives rise to a $\geq$ constraint in the dual. If a primal variable is restricted to be non-positive, the corresponding dual constraint is a $\leq$ inequality. Similarly, unrestricted primal variables lead to dual equalities. These results are summarized in the lower part of Table 4.1.

Table 4.1 may also be used to develop relationships for a minimization problem by reading the information from left to right. Suppose the objective is to minimize and a $\leq$ constraint is present. The corresponding dual variable in a maximization dual would be non-positive.

A second approach can also be followed; i.e., always transform the problem to standard form then write its dual. To illustrate, consider the LP problem

$$\begin{aligned} \text{Max} \quad & 3x_1 - 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_3 + 2x_3 = 20 \\ & -2x_1 + x_2 + x_3 \geq 10 \\ & x_1 \leq 0, \quad x_2 \geq 0, \quad x_3 - \ \text{unrestricted} \end{aligned}$$

Let us write the dual to this problem using the two approaches outlined above. First, let's convert the problem to standard form. To do this, the equality constraint must be replaced by two constraints:

$$x_1 + x_2 + 2x_3 \leq 20$$

$$x_1 + x_2 + 2x_3 \geq 20$$

In addition the second primal constraint should be multiplied through by -1, the first variable is replaced by its negative ($X_1 = - X_1^-$) and the third variable X_3 is replaced by $X_3^+ - X_3^-$. Making these substitutions and modifications gives

$$\begin{array}{lrcrcrcrcr}
\text{Max} & -3x_1^- & - & 2x_2 & + & x_3^+ & - & x_3^- & & \\
\text{s.t.} & -x_1^- & + & x_2 & + & 2x_3^+ & - & 2x_3^- & \leq & 20 \\
 & x_1^- & - & x_2 & - & 2x_3^+ & + & 2x_3^- & \leq & -20 \\
 & +2x_1^- & - & x_2 & - & x_3^+ & + & x_3^- & \leq & -10 \\
 & x_1^- & , & x_2 & , & x_3^+ & , & x_3^- & \geq & 0
\end{array}$$

The dual to this problem is

$$\begin{array}{lrcrcrcr}
\text{Min} & 20w_1 & - & 20w_2 & + & 10w_3 & & \\
\text{s.t.} & -w_1 & + & w_2 & + & 2w_3 & \geq & -3 \\
 & w_1 & - & w_2 & - & w_3 & \geq & -2 \\
 & 2w_1 & - & 2w_2 & - & w_3 & \geq & 1 \\
 & -2w_1 & + & 2w_2 & + & w_3 & \geq & -1 \\
 & w_1 & , & w_2 & , & w_3 & \geq & 0
\end{array}$$

Note that: a) the last two constraints can be rewritten as an equality, b) variables w_1 and w_2 can be combined into variable $w = w_1 - w_2$ which is unrestricted in sign, and c) we may substitute $w_3^- = -w_3$ and we may revise the inequality of the first constraint yielding

$$\begin{array}{lrcrcr}
\text{Min} & 20w & + & 10w_3^- & & \\
\text{s.t.} & w & - & 2w_3^- & \leq & 3 \\
 & w & + & w_3^- & \geq & -2 \\
 & 2w & + & w_3^- & = & -1 \\
 & w \ \text{unrestricted} & , & w_3^- & \leq & 0
\end{array}$$

which is the final dual.

This result can also be obtained from the use of the primal-dual relationships in Table 4.1. The primal objective is to maximize, so the dual objective is to minimize. The primal problem has 3 variables and 2 constraints, so the dual has 2 variables and 3 constraints. The first primal constraint is an equality, so the first dual variable is unrestricted in sign. The second primal constraint is a ($\geq$) inequality so the second dual variable should be non-positive. The first primal variable is restricted to be non-positive, so the first dual constraint is a ($\leq$) inequality X_2 is restricted to be non-negative, thus the second dual constraint is $\geq$; X_3 is unrestricted in sign. Thus, the third dual constraint is an equality. Then, the dual can be written as

$$
\begin{array}{llll}
Min & 20u_1 & + \; 10u_2 & \\
s.t. & u_1 & - \; 2u_2 & \leq \; 3 \\
 & u_1 & + \; u_2 & \geq \; -2 \\
 & 2u_1 & + \; u_2 & = \; 1 \\
 & u_1 \; unrestricted & , \; u_2 & \leq \; 0
\end{array}
$$

which if one substitutes $U_i = w_0$ and $U_2 = w_3^-$ is identical to that above.

4.4 The Character of Dual Solutions

If the primal problem possesses a unique nondegenerate, optimal solution, then the optimal solution to the dual is unique. However, dual solutions arise under a number of other conditions. Several cases which can arise are:

1) When the primal problem has a degenerate optimal solution, then the dual has multiple optimal solutions.
2) When the primal problem has multiple optimal solutions, then the optimal dual solution is degenerate.
3) When the primal problem is unbounded, then the dual is infeasible.
4) When the primal problem is infeasible, then the dual is unbounded or infeasible.

4.5 Degeneracy and Shadow Prices

The above interpretations for the dual variables depend upon whether the basis still exists after the change occurs. As shown in the previous chapter, there is a right hand side range over which the basis remains optimal. When a basic primal variable equals zero, the dual has alternative optimal solutions. The cause of this situation is generally that the primal constraints are redundant at the solution point and the range of right hand sides is zero. This redundancy means one does not need a full basic solution, so one of the basic variables is set to zero with the other basic variables likely to be nonzero. The best way to explain the implications of this situation is through an example. Consider the following problem

$$
\begin{array}{lccccc}
\text{Max} & 3X_1 & + & 2X_2 & & \\
 & X_1 & + & X_2 & \leq & 100 \\
 & X_1 & & & \leq & 50 \\
 & & & X_2 & \leq & 50 \\
 & X_1 & , & X_2 & \geq & 0
\end{array}
$$

Notice that at the optimal solution, $X_1 = 50$, $X_2 = 50$, the constraints are redundant. Namely, either the combination of the last two constraints or the first two constraints would yield the same optimal solution which is $X_1 = X_2 = 50$. The simplex solution of this problem shows a tie for the entering variable in the second pivot where one has the choice of placing X_2 into the solution replacing either the slack variable from the first or the third constraint. If the first slack variable (S_1) is chosen as basic then one gets $X_1 = 50$, $X_2 = 50$, $S_1 = 0$ while S_1 is basic. The associated shadow prices are 0, 3, and 2. On the other hand, if S_3 were made basic one gets $X_1 = 50$, $X_2 = 50$ $S_3 = 0$ with the shadow prices 2, 1, 0. Thus, there are two alternative sets of shadow prices, both of which are optimal. (Note, the dual objective function value is the same as the optimal primal in each case.)

The main difficulty with degeneracy is in interpreting the shadow price information. The shadow prices are taking on a direction (i.e., see the arguments in McCarl (1977)). Note that if one were to increase the first right hand side from 100 to 101 this would lead to a zero change in the objective function and X_1 and X_2 would remain at 50. On the other hand if one were to decrease that right hand side from 100 to 99 then one would obtain an objective function which is two units smaller because X_2 would need to be reduced from 50 to 49. This shows that the two alternative shadow prices for the first constraint (i.e., 0 and 2) each hold in a direction. Similarly if the constraint on X_1 was increased to 51, the objective function increases by one dollar as one unit of X_2 would be removed from the solution in favor of X_1; whereas, if the constraint was moved downward to 49, it would cost three dollars because of the reduction in X_1. Meanwhile, reducing the constraint on X_2 would cost two dollars, while increasing it would return to zero dollars. Thus in all three cases shadow prices take on a direction and the value of that change is revealed in one of the two dual solutions. This is quite common in degeneracy and may require one to do a study of the shadow prices or try to avoid degeneracy using a priori degeneracy resolution scheme as discussed in McCarl (1977); Paris (1991); Gal, and Gal et al., or as implemented automatically in OSL.

4.6 Primal Columns are Dual Constraints

One final comment relative to modeling is that the columns in the primal, form constraints on the dual shadow price information. Thus, for example, when a column is entered into a model indicating as much of a resource can be purchased at a fixed price as one wants, then this column forms an upper bound on the shadow price of that resource. Note that it would not be sensible to have a shadow price of that resource above the purchase price since one could purchase more of that resource. Similarly, allowing goods to be sold at a particular price without restriction provides a lower bound on the shadow price.

In general, the structure of the columns in a primal linear programming model should be examined to see what constraints they place upon the dual information. The linear programming modeling chapter extends this discussion.

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Table 4.1. Primal-Dual Correspondence for Problems not in standard form

Maximization Primal Problem	Minimization Dual Problem
Form of Primal Constraints	**Implication for Dual Variables**
$\sum_j a_{ij} X_j \le b_i$	$U_i \ge 0$
$\sum_j a_{ij} X_j = b_i$	U_i - unrestricted in sign
$\sum_j a_{ij} X_j \ge b_i$	$U_i \le 0$
Form of Primal Variables	**Implication for Dual constraints**
$X_j \ge 0$	$\sum_i a_{ij} U_i \ge c_j$
X_j - unrestricted in sign	$\sum_i a_{ij} U_i = c_j$
$X_j \le 0$	$\sum_i a_{ij} U_i \le c_j$