Contents

CHAPTER X: MODELING SUMMARY

Now that LP theory and basic modeling have been covered, a number of additional considerations involved with using models are covered, including variable and constraint types as well as LP assumptions.

10.1 Types of Constraints and Variables in Linear Programming Models

In most text books the LP problem is vastly oversimplified when first defined. For example, consider the problem

$$\begin{array}{lrcl} \text{Max} & CX & & \\ \text{s.t.} & AX & \leq & b \\ & X & \geq & 0 \end{array}$$

Where, the X's are defined as alternative production processes while the constraints (AX # b) are referred to as resource limitations. However, the previous chapters show there may be many different types of variables and constraints within such a formulation. This section develops a characterization of the various possible types of variables and constraints which can be used.

10.1.1 Types of Constraints

Possible constraint types include resource limitations, minimum requirements, supply-demand balances, ratio controls, upper/lower bounds, accounting relations, deviation constraints, and approximation or convexity constraints.

10.1.1.1 Resource Limitations

Resource limitations depict relationships between endogenous resource usage and exogenous resource endowments. A resource limitation restricts endogenous resource use to be less than or equal to an exogenous resource endowment. An example of a resource limitation constraint is

$$3X_1 \quad + \quad 4X_2 \quad \leq \quad 7$$

This constraint requires the sum of resources used in producing X_1, which uses 3 resource units per unit, plus those used in producing X_2, which uses 4 resource units per unit, to be no greater than an exogenous resource endowment of 7 units. Resource usage depends on the values of X_1

and X_2 determined by the model and thus is an endogenous quantity. This type of constraint appears in many of the formulations in Chapter 5, including the resource allocation problem.

10.1.1.2 Minimum Requirements

Minimum requirement constraints require an endogenously determined quantity to be greater than or equal to an exogenously specified value. A simple illustration is

$$X_1 \quad + \quad 2X_2 \quad \geq \quad 4$$

In this case the endogenous sum of X_1 plus two times X_2 is constrained to be greater than or equal to the exogenously imposed requirement of four. One may also express this constraint in less-than-or-equal-to form as

$$-X_1 \quad - \quad 2X_2 \quad \leq \quad -4$$

The minimum requirement often specifies that the model must meet exogenous demand through the endogenous supply of goods. This kind of constraint is present in many different types of programming models. An example appears in the transportation model of Chapter 5.

10.1.1.3 Supply and Demand Balance

The supply-demand balance requires that endogenous supply be balanced with endogenous demand. A typical example is

$$X_1 \quad \leq \quad X_2$$

This equation requires the endogenous demand for a good (X_1) to be less than or equal to the endogenous supply of that good (X_2). After moving all the variables to the left hand side, the constraint becomes

$$X_1 - X_2 \leq 0.$$

More generally, supply demand balances may involve exogenous quantities. Consider the inequality

$$2X_1 - X_2 \leq 3.$$

Here, the difference between endogenous demand ($2X_1$) and supply (X_2) is less than or equal to an exogenous supply of 3 units. This inequality can also be expressed in the following form:

$$2X_1 \leq X_2 + 3$$

which says that the endogenous demand ($2X_1$) must be less than or equal to total supply, which consists of endogenous supply (X_2) plus exogenous supply (3). A related situation occurs under the constraint

$$X_1 - 4X_2 \leq -2.$$

Here, the difference between endogenous supply and endogenous demand is less than or equal to minus 2. This can be rewritten as

$$X_1 + 2 \leq 4X_2$$

which states that endogenous demand (X_1) plus exogenous demand (2) is less than or equal to endogenous supply ($4X_2$). In general, supply-demand balances are used to relate endogenous supply and demand to exogenous supply and demand. The general case is given by

$$\text{Demand}_{En} + \text{Demand}_{Ex} \leq \text{Supply}_{En} + \text{Supply}_{Ex}\ .$$

Here, the sum of demand over endogenous and exogenous sources (respectively denoted by the subscripts En and Ex) must be less than or equal to the supply from endogenous and exogenous sources. Manipulating the endogenous variables to the left hand side and the exogenous items to the right hand side gives

$$\text{Demand}_{En} - \text{Supply}_{En} \leq \text{Supply}_{Ex} - \text{Demand}_{Ex}\ .$$

Here endogenous demand minus endogenous supply is less than or equal to exogenous supply minus exogenous demand.

This constraint contains the resource limitation and minimum requirement constraints as special cases. The resource limitation constraint exhibits zero endogenous supply and exogenous demand. The minimum requirement constraint exhibits zero endogenous demand and exogenous supply.

Supply demand balances are present in many of the examples of Chapter 7. The assembly, disassembly, assembly - disassembly, and the sequencing problems all possess such constraints.

10.1.1.4 Ratio Control

Ratio control constraints require the ratio of certain endogenous variables to be no more than an endogenous sum, possibly influenced by exogenous factors. Specifically suppose that a number of units of X_1 have to be supplied with every unit of X_2. For example, a LP formulation of an automobile manufacturer might require a constraint to insure that there are four tires for every car sold. Such a situation would be modeled by

$$4X_2 - X_1 \leq 0$$

where X_1 is the number of tires and X_2 the number of cars sold. In order for one unit of X_2 to be sold, 4 units of X_1 must be supplied.

The general case is depicted by

$$EN_{rat} \leq p\,(w_{EN}\ EN_{rat} + EN_{other} + EX_{other}).$$

where the left hand side elements are denoted with the subscript "rat," and the right hand side elements with "other." EN denotes endogenous variables and EX denotes exogenous constants. The parameter w_{EN} is nonzero only when the endogenous variables (EN_{rat}) are part of the right hand side. The constraint requires that the endogenous "rat" expression be less than or equal to p times the sum of the "rat" term or variables plus the "other." Manipulating this constraint so that all the endogenous variables are on the left hand side gives

$$(1 - pw_{EN})EN_{rat} - p\, EN_{other} \leq p\, EX_{other}$$

This expression is rather abstract and is perhaps best seen by the example. Suppose we wish the variable X_1 to be no more than 25 percent of $X_1 + X_2$. Thus

$$X_1 \leq .25\,(X_1 + X_2)$$

Placing all the endogenous variables on the left hand side yields

$$.75\,X_1 - .25\,X_2 \leq 0$$

Consider another example which includes exogenous factors. Suppose that

$$(X_1 + 3) \leq .25\,(2\,X_1 + 3\,X_2 + 10)$$

this can be written as

$$.50\,X_1 - .75\,X_2 \leq .5$$

Here we have a requirement between X_1 and X_2 and an exogenous constant appearing on the right hand side.

Finally, if the endogenous variables do not appear on the right hand side (for example, where X_1 is less than or equal to one-third the sum of $X_2 + 4X_3$) then the inequality would be manipulated to state:

$$X_1 - 1/3\,X_2 - 4/3\,X_3 \leq 0$$

This is an example where the w's in the ratio control constraint are zero.

This particular constraint type is a special case of the supply/demand balances. It is not used explicitly in any of the general formulations, but would also be used in a feed problem formulation where the quantity of feed to be produced was not exogenously given (i.e., on the right hand side) but rather was an endogenous variable.

10.1.1.5 Bounds

Upper and lower bounds have important implications for the performance of the simplex algorithm. Upper bounds are resource limitation constraints; however, they only involve a single variable. Similarly, lower bounds are minimum requirement constraints on a single variable. Examples are

$$X_1 \leq 4$$

$$X_1 \geq 2.$$

Such constraints are usually exploited by LP solvers so that they do not enter the basis inverse.

10.1.1.6 Accounting Relations

Accounting relations are used to add endogenous sums for model solution summary purposes. These relations are used for modeler convenience in summarizing a solution (i.e., adding up total labor utilized by crop). Accounting relations can be depicted as either

$$\sum_{j=1}^{n} A_{ij} X_j \geq 0 \qquad \text{or} \qquad \sum_{j=1}^{n} A_{ij} X_j \quad S = 0$$

In the first case the surplus variable would equal the sum of AX (assuming AX is always non-negative). The second form of the equation simply introduces an accounting variable which takes on the value of the sum. Accounting relations are discussed in the purposeful modeling section.

10.1.1.7 Deviation Constraints

Deviation constraints are used to develop the endogenous deviation of a particular sum from a target level. The general format of these constraints is as follows:

$$\sum_{j} g_{ij} x_j + Dev_i = T_i$$

Here T_i is a target level and Dev_i is a deviation variable indicating the amount the endogenous sum

(3 $g_{ij}x_j$) deviates (as measured by the deviation variable Dev_i) from the target level (T_i). The deviation constraint concept is utilized in the nonlinear transformations involving absolute value, multi-objective programming, and risk modeling.

10.1.1.8 Approximation or Convexity Constraints

A convexity constraint requires the sum of a set of variables to be equal to or possibly less than or equal to one. These are commonly used in approximations such as those under the separable programming section of the nonlinear approximations chapter.

10.1.2 Types of Variables

There are many different types of variables. Production, sales, purchase, transformation, slack, surplus, artificial, step, deviation and accounting variables are discussed in this section.

10.1.2.1 Production Variables

Production variables depict the production of outputs from inputs. Such a variable is represented by

X_2 in the LP problem

$$
\begin{array}{llccccccccccccccl}
(1) & \text{Max} & aX_1 & & & - & dX_3 & - & gX_4 & - & iX_5 & & & - & mX_7 & - & nX_8 & + & qX_9 & & \\
(2) & \text{s.t.} & X_1 & - & bX_2 & & & & & & & & & & & & & & & \leq & 0 \\
(3) & & & & cX_2 & - & X_3 & & & + & X_5 & & & & & & & & & \leq & 0 \\
(4) & & & & eX_2 & & & & & - & X_5 & & & & & & & & & \leq & f \\
(5) & & & & hX_2 & & & - & X_4 & & & & & & & & & & & \leq & 0 \\
(6) & & & & jX_2 & & & & & & & - & X_6 & & & & & & & = & 0 \\
(7) & & & & kX_2 & & & & & & & & & - & X_7 & + & X_8 & & & = & T \\
(8) & & & & -pX_2 & & & & & & & & & & & & & + & X_9 & \leq & 0 \\
(9) & & & & rX_2 & & & & & & & & & & & & & & & \leq & b \\
 & & & & & & & & & & & & & & & & & & X_i & \geq & 0
\end{array}
$$

Note that X_2 produces items which are transferred into the equations (2) and (8). The X_2 variable also uses inputs from equations (3) and (5) and utilizes a fixed resource which is represented by (4). Thus, X_2 depicts a multi-factor/multi-product production process. Production coefficients do not always explicitly appear in the constraint equations; rather, production may simply yield revenue in the objective function as in the resource allocation and sequencing problems. Production activities may also use inputs which have pre-specified costs, thus the objective function coefficients may involve revenue and/or cost terms. The purposeful modeling section provides such an example.

10.1.2.2 Sales Variables

Sales variables reflect the sale of an item at an exogenously determined price. For example, variable X_1 in the above tableau depicts the sale of an item at price a, where the item is drawn from the supply-demand balance that relates X_1 to the production activity X_2 (equation (2). X_9 is also a sales variable. Sales variables appear in numerous examples above. For example, see the assembly- disassembly and joint product formulations.

10.1.2.3 Purchase Variables

Purchase variables depict the purchase of items at exogenously specified prices with the items made available for use within the model. Examples of this type of variable are X_3 and X_4 above. For example, one unit of X_3 yields one unit of supply to the supply-demand balance equation (3) and enters the objective function with a coefficient of -d. Purchase activities are illustrated in the assembly formulation.

10.1.2.4 Transformation Variables

Transformation variables transform the location, time availability, unit or form characteristics of an item (although other inputs may be required to do this). Examples of such variables include transportation variables which alter location, storage variables which alter time availability, unit transformation variables that convert the units from, say tons to pounds, or variables which transform a good from one form to another, possibly with the addition of other inputs. An example of this type of variable includes beef slaughter, where pounds of beef on the hoof are converted into hanging carcass beef.

The variable X_5 in the LP example given by (1) - (9) is a transformation variable depicting transformation at per unit cost I of the resources in constraint (3) into the resources in constraint (4). Transformation variables appear in the storage and transportation examples.

10.1.2.5 Slack Variables

Slack variables represent the amount of excess resources (i.e., resources which are unused in production). Ordinarily, they have a zero objective function coefficient and a plus one entry in a single constraint. Slack variables are defined in association with less than or equal to constraints representing the extent to which the endogenous quantity is less than the right hand side. Slack variables do not play a large role in model formulations (although deviation and accounting variables are forms of slack variables). However, slack variables can play an important role in solution interpretation. Modelers should check which resources are left unused (with non-zero slack) and question whether such a situation is reasonable.

10.1.2.6 Surplus Variables

Surplus variables are analogous to slack variables; they have zero objective function coefficients and a coefficient only in one particular row. They represent the amount that the left hand side of a constraint is greater than the right hand side. Surplus variables do not ordinarily play a large role in applied modeling. However, they may be important in the interpretation of the solution of a model. For example, the magnitude of a surplus variable may indicate the extent to which over-production occurs above a minimum requirement.

10.1.2.7 Artificial Variables

Artificial variables are most often utilized to make an infeasible problem feasible, allowing the violation of equality constraints or minimum requirements. Artificial variables ordinarily have a large cost in the objective function and a coefficient in the particular row with which they are associated. However, artificial variables can play a role in applied modeling. For example, artificial variables can be used to prohibit an infeasible solution from arising in solvers. Artificial variables also play an important role in discovering the causes of infeasibilities, as discussed in the chapter on debugging models.

10.1.2.8 Step Variables

Linear programs may involve the approximation of nonlinear phenomena. Step variables are often used in such approximations. One may, for example, utilize step variables to represent different portions of an increasing cost function. Step variables receive their name from their portrayal of nonlinear functions as a series of piece-wise linear steps. Step variables appear in the separable programming formulations.

10.1.2.9 Deviation Variables

Deviation variables tell the amount by which an endogenous sum deviates from a target value. Such variables are illustrated in the LP model given by (1)-(9) by X_7 or X_8. For example, in equation (7), these variables indicate the amount kX_2 deviates from the target value T. The variable X_7 gives the amount that the sum is over the target while the variable X_8 gives the amount the sum is under the target. These variables are analogous to surplus and slack variables; however, they may have an objective function coefficient which reflects costs or revenues associated from "missing" the target. These variables will work properly as long as the objective function is properly structured as explained in the multi-objective programming chapter. Deviation variables are also an important part in the LP approaches to regression (as used in the absolute value formulation) and in the MOTAD formulation.

10.1.2.10 Accounting Variables

An accounting variable is typically used to indicate the value of endogenous sums so that the analyst need not manually summarize the solution. The variable X_6 in equation (6) is an example of this type of variable. These variables are also prominently featured in the section on purposeful modeling.

10.2 "Violations" of the Algorithmic Assumptions

The algorithmic assumptions of LP hold for individual variables within a linear program but not necessarily for the total process represented. Thus, modeling techniques can be used to generate formulations which, for practical purposes, invalidate the algorithmic assumptions. Let us consider models which nominally appear to violate each of the algorithmic assumptions.

10.2.1 Nonproportional Example

It is possible to satisfy the algorithmic assumptions regarding proportionality while formulating nonproportional problems. For example, suppose a production process exhibits diminishing returns to scale (i.e., doubling the level of inputs does not double the output). This may be modeled as follows:

$$
\begin{array}{llrrrrl}
\text{Max} & 10Y & & & -Z_1 & -8Z_2 & \\
\text{s.t.} & Y & -6X_1 & -1.2X_2 & & & \leq 0 \\
 & & 3X_1 & +3X_2 & -Z_1 & & \leq 0 \\
 & & 4X_1 & +4X_2 & & -Z_2 & \leq 0 \\
 & & X_1 & & & & \leq 4 \\
 & & & X_2 & & & \leq 4 \\
 & Y, & X_1, & X_2, & Z_1, & Z_2, & \geq 0
\end{array}
$$

In this model, a single output Y is produced from two production processes depicted by X_1 and X_2 with X_1 and X_2 upper bounded at four. The production processes utilize two inputs denoted by Z_1 and Z_2. Variable X_1 uses three units of the first input and four units of the second input and produces six units of the output Y. Variable X_2 uses the same mix of inputs, but produces 1.2 units of output which is one-fifth the amount produced by X_1. When inputs are used in the combination 4 units of the second input to 3 of the first, then for any combination between zero and 12 units of the first input (along with 16 units of the second), six units of output are produced per $3Z_1$ and $4Z_2$ used in combination. However, after using 12 units of Z_1 and $16Z_2$, the production process X_2 must be used yielding a marginal product of 1.2 units of production for the inputs used in the same proportion. In this example, doubling the level of input usage does not result in a doubling of output, but rather in only a 20 percent increase.

Are the algorithmic assumptions violated? Yes and no. They are not mathematically violated but they are conceptually violated. The assumptions hold for the individual activities, for example, going from $X_1 = .5$ to $X_1 = 1$ would involve the doubling of the inputs, and a doubling of outputs. However, because of the upper bound constraint on X_1, the solution $X_1 = 4$ is feasible, the solution $X_1 = 8$ is not. Consequently, the model must use X_2 yielding less output per unit of input.

In general, the proportionality assumption can be relaxed using multiple variables. The joint product section of chapter 7, as well as the separable programming and nonhomogeneous of degree one sections of chapter 9 provide further examples. Formal relaxation of this assumption is done through a number of techniques including integer, quadratic, and nonlinear programming. A reconciliation of LP modeling with the concept of diminishing returns is presented in the separable programming sections.

10.2.2 Non-Additive Example

Models may also be constructed which appear to violate the additivity assumption. Suppose a production process involves two inputs which can be substituted in production. This may be modeled as follows:

$$\begin{array}{llccccccl} \text{Max} & 3Y & & & & & & & \\ \text{s.t.} & Y & - & 2X_1 & - & 2X_2 & - & 2X_3 & \leq 0 \\ & & & 4X_1 & + & 2X_2 & + & X_3 & \leq r_1 \\ & & & X_1 & + & 2X_2 & + & 4X_3 & \leq r_2 \\ & Y, & & X_1, & & X_2, & & X_3 & \geq 0. \end{array}$$

Note this formulation depicts the production of Y using production processes X_1, X_2, or X_3. Each process produces 2 units of Y; however, inputs are used in different proportions. X_1 uses four units of input 1 and one unit of input 2; X_2 utilizes equal combinations of the two inputs, while X_3 uses one unit of input 1 with four units of input 2. The formulation is constrained by input availability where the quantity inputs available are designated as r_1 and r_2.

Now let us illustrate the nonadditive nature of this formulation. Suppose equal amounts of the inputs are available ($r_1 = r_2$), then it would be optimal to produce in a pattern utilizing the inputs in equal proportions. Note that by producing X_1 and X_3 in equal amounts, the inputs would be used in equal proportion, i.e., setting both variables to one would produce 1.6 units of output while utilizing 2 units of inputs of r_1 and r_2. Thus, 1.6 units of output are attained when using 0.4 units of each variable. However, when activity 2 is utilized at least two units of output are produced when using two units of each input. Total input usage is the same in both cases, however, more production arises out of the second production process then by adding the output of the first and third process. Thus, we get more out of using the inputs together, f(X + Y), than we do separately, f(X) + f(Y).

Does this violate the algorithmic assumptions? Within the model the production processes are

strictly additive. Combination of any group of X's leads to an additive output effect. However, by utilizing different variables, a production process may be represented which is not strictly additive. Thus, one can usually handle nonadditive cases between variables by including "better" variables which are more productive (i.e., X_2 above). Nevertheless, the additivity assumption always holds for the individual variables. It may not hold for the model through the combination of variables. This assumption is formally relaxed by the models covered in the nonlinear, price endogenous and risk chapters.

10.2.3 Uncertainty Examples

The certainty assumption may also be relaxed. Suppose we model a production process involving, a cost of \$3 in period 1 but that, in the second time period we are uncertain about how much of the product will be produced (e.g., harvested). Suppose that one of two uncertain events can occur in the second time period: no more than 2 units of the product may be sold for a price of \$5.00 with a probability of .3 <u>or</u> no more than 3 units of product could be sold at a price of \$4 with a probability of .7. This problem may be formulated as a classical so-called two-stage optimization problem (Dantzig, 1955). The formulation is

$$\begin{array}{llccccl} \text{Max} & -3Y & + & .3(5X_1) & + & .7(4X_2) & \\ \text{s.t.} & -Y & + & X_1 & & & \leq 0 \\ & & & X_1 & & & \leq 2 \\ & -Y & & & + & X_2 & \leq 0 \\ & & & & & X_2 & \leq 3 \\ & Y, & & X_1, & & X_2 & \geq 0. \end{array}$$

In this formulation a certain cost of $3 is incurred when using variable Y. In turn, the production of Y permits sale under the two probabilistic events. The amounts sold are denoted X_1 or X_2 depending upon the event. Resources cannot be shifted between X_1 and X_2 (i.e., they are two mutually exclusive states of nature), thus, there are independent limits on the sale of X_1 and X_2. However, Y precedes both. The objective function reflects the maximization of expected profits which are the expected revenue from sales less the cost of Y.

Thus, this formulation explicitly includes uncertainty. But, is the certainty assumption violated? Again, this formulation simultaneously satisfies and violates the algorithmic assumptions of LP. We have incorporated uncertainty within the formulation, but each variable contains certain coefficients. However, the overall model represents production under uncertainty. The uncertainty problem has been expressed in a problem where the model is certain of the uncertainty. Additional certainty assumption relaxations are discussed in the risk chapter. The specific example above is a sequential uncertainty, discrete stochastic or two-stage stochastic programming with recourse problem.

10.2.4 Noncontinuous Example

The continuity assumption when violated involves decision variables which are integer valued by nature (i.e., the number of cows, for instance). This maybe relaxed by rounding when in the optimal solution

the integer variables have very large values. A problem of this type is as follows:

$$\begin{array}{llcllcl} \text{Max} & 3X_1 & + & 2X_2 & & \\ \text{s.t.} & X_1 & + & X_2 & \leq & 10{,}000 \\ & 2X_1 & - & X_2 & \leq & 4{,}000 \\ & X_1, & & Y_2 & \geq & 0 \text{ and integer} \end{array}$$

The solution without the requirement that X_1 and X_2 be integer is X_1 = 4,666 2/3 and X_2 = 5,333 1/3. The model user might be willing to round this solution interpreting the solution as producing 4,667 of the first product and 5,333 of the second. This would clearly not be the optimal solution but might be practical and "close enough." Note, however, that the answer 4,667 and 5,333 slightly violates the second constraint. Nevertheless, decision makers might be willing to adopt this solution. In a practical problem this answer might even be interpreted as 4,700 and 5,300.

The continuity assumption is not practically relaxed other than by rounding large solution values or by solving an integer programming problem.

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