13 CHAPTER XIII: PRICE ENDOGENOUS MODELING

13.1 Introduction

A common economic application of nonlinear programming involves price endogenous models. In the standard LP model, input and output prices or quantities are assumed fixed and exogenous. Price endogenous models are used in situations where this assumption is felt to be untenable. Such problems can involve modeling an industry or sector such that the level of output or purchases of inputs is expected to influence equilibrium prices.

$$P_d = a_d - b_d Q_d .$$

The approach to formulating such problems was motivated by Samuelson, who suggested solving optimization problems whose first-order conditions constituted a system of equations characterizing an equilibrium. Suppose we follow this approach by first defining a system of equations, then posing the related optimization problem. Let an inverse demand equation be defined where P_d is price of the product, a_d is the intercept, b_d is the slope, and Q_d is the quantity demanded. Similarly, suppose we have an inverse supply equation

$$P_s = a_s + b_s Q_s ,$$

where the terms are defined analogously. An equilibrium solution would have price and quantity equated and would occur at the simultaneous solution of the equations

$$P_d = P_s$$

or

$$a_d - b_d Q_d = a_s + b_s Q_s$$

and

$$Q_d = Q_s$$

One should also recognize some possible peculiarities of the equilibrium, namely it is possible that the markets could clear at zero quantity, in which case the supply price might be greater than or equal to the demand price.

Thus, we can write the condition that the equilibrium price (P^*) is greater than or equal to the demand price

$$a_d - b_d Q_d \leq P^*$$

Simultaneously, the market price may be less than the supply price,

$$a_s + b_s Q_s \geq P^*$$

One can also argue that these two relations should only be inequalities when the quantity supplied or demanded equals zero. Namely, when the price of demand is less than the equilibrium price, then zero quantity should be demanded. Similarly, when the price of supply is greater than the equilibrium price, then zero quantity should be supplied. Simultaneously, when a non-zero quantity is supplied or demanded, then the equilibrium price should equal the supply or demand price. This relationship can be expressed through complementary slackness like relations where

$$\left(a_d - b_d Q_d - P^*\right)Q_d = 0$$
$$\left(a_s + b_s Q_s - P^*\right)Q_s = 0$$

One should also recognize that the quantity supplied must be greater than or equal to the quantity demanded

$$Q_s \geq Q_d$$

but, if the quantity supplied is strictly greater than the quantity demanded, then the equilibrium price should be zero. Mathematically this relationship is

$$\left(-Q_s + Q_d\right)P^* = 0$$

Finally, we state nonnegativity conditions for price and quantities,

$$Q_d, \quad Q_s, \quad P^* \quad \geq \quad 0.$$

The above equations are similar to the Kuhn-Tucker conditions. In particular, if P^* is taken to be a dual variable, then the above equation system is equivalent to the Kuhn-Tucker conditions of the following optimization model

$$
\begin{array}{llllllllll}
\text{Max} & a_d & Q_d & - & 1/2 & b_d & Q_d^2 & - & a_s & Q_s & - & 1/2 & b_s & Q_s^2 & & \\
\text{s.t.} & & Q_d & & & & & - & & Q_s & & & & & \leq & 0 \\
 & & Q_d, & & & & & & & Q_s & & & & & \geq & 0
\end{array}
$$

where P^* is the dual variable associated with the first constraint. Optimizing this model solves our equilibrium problem.

This is a quadratic programming problem. The formulation was originally motivated by Enke; and Samuelson. Later it was fully developed by Takayama and Judge (1973). The general form maximizes the integral of the area underneath the demand curve minus the integral underneath the supply curve (Figure 13.1), subject to a supply-demand balance. The resultant objective function value is commonly called consumers' plus producers' surplus.

The graphical representation allows one to develop a practical interpretation of the shadow price. Consider what happens if the $Q_d - Q_s \leq 0$ constraint is altered so that the right hand side is one ($Q_d - Q_s \leq 1$). In this case demand is allowed to be one unit greater than supply. Assuming the one unit is small relative to total quantity then we get an area increment that is approximately the height of the equilibrium price and one unit wide (Figure 13.2). The resultant objective function then is the original value plus an area equaling the equilibrium price. Thus, the change in the objective function when increasing the right hand side (the shadow price) can be interpreted as the equilibrium price. This also equals the Lagrange multiplier introduced when applying Kuhn-Tucker theory.

13.1.1 Example

Suppose we have

$$
\begin{array}{lllll}
P_d & = & 6 & - & .3Q_d \\
P_s & = & 1 & + & .2Q_s
\end{array}
$$

Then the formulation is

$$
\begin{array}{llllllll}
\text{Max} & 6Q_d & - & 0.15Q_d^2 & - & Q_s & - & 0.1Q_s^2 & & \\
 & Q_d & & & - & Q_s & & & \leq & 0 \\
 & Q_d, & & & & Q_s & & & \geq & 0
\end{array}
$$

The GAMS formulation of this model is in Table 13.1 and file PRICEND. Note that there are two important changes in this setup compared to an LP. The first is that the objective function equation contains the nonlinear squared terms. The second is that in the SOLVE statement we indicate that the problem is a nonlinear programming problem by saying SOLVE USING NLP. The solution to the model is given in Table 13.2. It indicates that the quantity supplied and demanded equal 10, that the price is 3 (equaling the shadow price on the commodity balance row), and that consumers' plus producers' surplus equals 25.

The above example is a simple case where we have a single supply and single demand curve. Clearly, no one would solve this problem using nonlinear programming, as it could be easily solved by hand. However, the problem does illustrate the formulation of price endogenous models.

13.2 Spatial Equilibrium

A common price endogenous model application involves the spatial equilibrium problem. This problem is an extension of the transportation problem relaxing the assumption of fixed supply and demand. The problem is motivated as follows. Production and/or consumption usually occurs in spatially separated regions, each of which have supply and demand relations. In a solution, if the regional prices differ by more than the interregional cost of transporting goods, then trade will occur and the price difference will be driven down to the transport cost. Modeling of this situation addresses the questions of who will produce and consume what quantities and what level of trade will occur.

Takayama and Judge (1973) developed the spatial equilibrium model to deal with such situations. Suppose that in region i the demand for the good of interest is given by

$$P_{di} = f_i\left(Q_{di}\right)$$

where p_{di} is the demand price in region i while Q_{di} is the quantity demanded. Simultaneously suppose the supply function for region i is

$$P_{si} = s_i\left(Q_{si}\right)$$

where p_{si} is the supply price in region i, and Q_{si} the quantity supplied. A "quasi-welfare function" for each region can be defined as the area between the supply and demand curves,

$$W_i\left(Q^*_{si},\quad Q^*_{di}\right) = \int_0^{Q^*_{di}} P_{di} dQ_{di} \ - \int_0^{Q^*_{si}} P_{si} dQ_{si}.$$

The total welfare function across all regions is the sum of the welfare functions in each region less total transport costs. Suppose T_{ij} represents the amount of good shipped from i to j at cost c_{ij}. Then the net welfare is

$$NW = \sum_i W_i\left(Q_{di}, Q_{si}\right) - \sum_i \sum_j c_{ij} T_{ij}$$

In turn we may form an optimization problem with the NW expression as the objective function plus the constraints from the transportation model. These constraints involve a demand balance requiring that incoming shipments to a region be greater than or equal to regional demand,

$$Q_{di} \le \sum_j T_{ji} \quad \text{for all i}$$

and a supply balance requiring that outgoing shipments do not exceed regional supply

$$Q_{si} \ge \sum_j T_{ij} \quad \text{for all i.}$$

The resultant problem becomes

$$
\begin{array}{llllll}
\text{Max} & \sum_i \left(\int_0^{Q_{di}^*} P_{di} \, dQ_{di} - \int_0^{Q_{si}^*} P_{si} \, dQ_{si} \right) - \sum_i \sum_j c_{ij} T_{ij} & & & & \\
\text{s.t.} & Q_{di} - \sum_j T_{ji} & \leq & 0 & \text{for all} & i \\
 & -Q_{si} + \sum_j T_{ij} & \leq & 0 & \text{for all} & i \\
 & Q_{di}, \quad Q_{si}, \quad T_{ij} & \geq & 0 & \text{for all} & i \text{ and } j
\end{array}
$$

This problem yields an equilibrium solution as long as the demand curves are downward sloping and the supply curves are upward sloping. The nature of the solution and the equilibrium can best be revealed by investigating relevant parts of the Kuhn-Tucker Conditions.

$$
\begin{array}{lllll}
\frac{\partial L}{\partial Q_{di}} = P_{di} - \lambda_{di} \leq 0 & \left(\frac{\partial L}{\partial Q_{di}} \right) Q_{di} = 0 & Q_{di} \geq 0 \\
\frac{\partial L}{\partial Q_{si}} = -P_{si} + \varphi_{si} \leq 0 & \left(\frac{\partial L}{\partial Q_{si}} \right) Q_{si} = 0 & Q_{si} \geq 0 \\
\frac{\partial L}{\partial T_{ij}} = -c_{ij} + \lambda_{dj} - \varphi_{si} \leq 0 & \left(\frac{\partial L}{\partial T_{ij}} \right) T_{ij} = 0 & T_{ij} \geq 0
\end{array}
$$

These conditions imply that the shadow price in region i on the first constraint set ($\square_{di}$), assuming Q_{di} is positive, equals the demand price while the second shadow price Ψ_{si} equals the supply price if Q_{si} is positive. The transportation activities insure that the demand price in a region must be less than the supply prices in all other regions plus transport costs.

The solution to this problem yields the level of supply by region (Q_{si}), the level of consumption by region (Q_{di}), and the level of trade between regions ($T_{i,j}$ $i \neq j$) as well as the level of internal consumption (T_{ii}). Price in each region is found in the dual variables.

The relationships between the equilibrium prices can take on one of several cases. Namely: a) if region i fills some of its own demand (i.e. $T_{ii} > 0$), then the domestic supply and demand prices are equal; b) if region i exports to region j, ($T_{ij} > 0$), then the demand price in region j equals the supply price in region i plus transport cost; c) if region j does not export to region i, then generally $P_{dj} < P_{si} + C_{ij}$ indicates trade is not desirable since the price differential will not support the transport cost.

In this problem, the variable T_{ii} represents the quantity produced in region i and consumed in that region. For example, suppose there are 3 regions, then total supply in region 1 is denoted by Q_{s1}. Total exports to region 2 and region 3 are $T_{12} + T_{13}$. The amount produced in region 1 and not exported, thus locally consumed, is

$$Q_{s1} - T_{12} - T_{13} = T_{11}.$$

In inequality form, the balance is

$$Q_{s1} \geq T_{11} + T_{12} + T_{13}$$

The spatial equilibrium literature commonly deals with a special case of this problem namely the case where the supply and demand functions are both linear, i.e.,

$$p_{si} = a_i + b_i\ Q_{si}\text{, and }\ p_{di} = e_i - f_i\ Q_{di}$$

In this case the objective function is quadratic and becomes:

$$\text{Max}\ \sum_i (a_i Q_{di} - 1/2\, b_i\ Q_{di}^2 - e_i Q_{si} - 1/2\, f_i\ Q_{si}^2) - \sum_i \sum_j c_{ij} T_{ij}$$

13.2.1 Example

Suppose we have three entities (US, Europe, Japan) trading a single homogeneous commodity. Suppose supply curves are present only in the US and Europe and the parameters of these curves are

$$\begin{aligned} P_{s,U} &= 25 + Q_{s,U} \\ P_{s,E} &= 35 + Q_{s,.E} \end{aligned}$$

while the demand curves are

$$\begin{aligned} P_{d,U} &= 150 - Q_{d,U} \\ P_{d,E} &= 155 - Q_{d,E} \\ P_{d,j} &= 160 - Q_{d,J} \end{aligned}$$

and internal transport is free. Also suppose transport between the US and Europe costs 3 in either direction, while it costs 4 between the US and Japan and 5 between Europe and Japan. The formulation of this problem is

$$\begin{array}{lllllll} \text{Max} & 150Q_{d,U} & - 1/2Q_{d,U}^2 & + 155Q_{d,E} & - 1/2Q_{d,E}^2 & + 160Q_{d,J} & - 1/2Q_{d,J}^2 \\ & -25Q_{s,U} & - 1/2Q_{s,U}^2 & - 35Q_{s,E} & - 1/2Q_{s,E}^2 & & \\ & -0\,T_{U,U} & - 3\ T_{U,E} & - 4\ T_{U,J} & - 3\ T_{E,U} & - 0\ T_{E,E} & - 5\ T_{E,J} \\ & -4\,T_{J,U} & - 5\ T_{J,E} & & & & \end{array}$$

$$\begin{array}{lllllllllllll} \text{s.t} & Q_{d,J} & & & & & -T_{U,U} & & & -T_{E,U} & & & \le 0 \\ & & Q_{d,E} & & & & & -T_{U,E} & & & -T_{E,E} & & \le 0 \\ & & & Q_{d,J} & & & & & -T_{U,J} & & & -T_{E,J} & \le 0 \\ & & & & -Q_{s,U} & & +T_{U,U} & +T_{U,E} & +T_{U,J} & & & & \le 0 \\ & & & & & -Q_{s,E} & & & & +T_{E,U} & +T_{E,E} & +T_{E,J} & \ge 0 \\ & Q_{d,U}, & Q_{d,E}, & Q_{d,J}, & Q_{s,U}, & Q_{s,E}, & T_{U,U}, & T_{U,E}, & T_{U,J}, & T_{E,U}, & T_{E,E}, & T_{E,J} & \ge 0 \end{array}$$

The solution to this problem yields an objective function value of 9193.6. The optimal values of the variables are shown in Table 13.3.

This solution indicates consumption of 45.4 units in the U.S., and 51.4 in both Europe and Japan, while 79.6 units are supplied in the US and 68.6 in Europe. The U.S. and Europe both get all of their consumption quantities from domestic production while the U.S. exports 34.2 units to Japan and Europe exports 17.2. The equilibrium prices appear in the shadow price column. The price in the U.S. is 104.6 while the European price is 103.6. Note the Japanese price is 108.6 which is higher than the price in the other two regions by the transport cost. These prices may also be recovered by plugging the equilibrium quantities into the demand and supply curves.

The utility of this model may be demonstrated by performing some slight extensions. Suppose we use the model to examine the costs and effects of trade barriers and their cost. Specifically consider model solution a) without any trade, b) with the U.S. imposing a quota of 2 units, and c) with the U.S. imposing a 1 unit export tax while Europe imposes a 1 unit export subsidy. Modeling these items involves the addition of a constraint which limits exports from a region to zero, the quota or an infinite amount depending on the scenario and the alteration of the inter-country transport costs to reflect the subsidy/tax. The GAMS file SPATEQ shows the implementation of these features. Table 13.4 presents a summary of the case solutions. Note that the expected results occur. Without trade domestic consumers in the U.S. and Europe receive cheaper prices and consume more, but Japanese consumers receive nothing. Simultaneously U.S. and European producers supply less and receive lower prices. Under the U.S. quota, a trade realignment occurs. This also happens under the tax and subsidy scenario. One may also interpret the objective function in terms of welfare and could decompose it to reveal the welfare positions of the consumers and producers by country. All in all, this example illustrates the potential usefulness of the spatial equilibrium, price endogenous structure.

13.3 Multi-Market Case

The price endogenous model may also be extended to multiple markets and products. Suppose one wishes to construct a model with multiple sources of supply and demand. For a single homogenous commodity let us illustrate this with an example.

Suppose we have two sources of wheat supply (domestic and import) and three sources of demand (wheat for making bread, wheat for making cereal, and wheat for export). Define Q_d and Q_i as the quantities supplied by domestic and import sources, respectively, with P_{sd} and P_{si} as the corresponding prices. Further, define X_b, X_c and X_e as the quantities of bread, cereal and exported wheat demanded, with P_{db}, P_{dc} and P_{de} the accompanying prices. Let the supply and demand curves be dependent on own price only with the supply curves:

domestic supply: $P_{sd} = 2.0 + 0.003 \; Q_d$

import supply: $Psi = 3.1 + 0.0001 \; Q_{i*}$

and the demand curves:

bread demand: $P_{db} = 0.75 - 0.0004 \; X_b,$

cereal demand: $P_{dc} = 0.80 - 0.0003 \; X_c,$

export demand: $P_{de} = 3.40 - 0.0001 \; X_e.$

Also suppose that one bushel (bu.) of wheat yields 5 units of bread, 6 units cereal or 1 exported bushel.

A problem which depicts this situation is

$$
\begin{array}{llllllllllll}
\text{Max} & (0.75-1/2*.0004X_b)X_b & + & (0.8-1/2*.0001X_c)X_c & + & (3.4-1/2*.0001X_e)X_e & - & (2.0+1/2*.003Q_d)Q_d & - & (3.1+1/2*.0001Q_i)Q_i & & \\
\text{s.t.} & 1/5X_b & + & 1/6X_c & + & X_e & - & Q_d & - & Q_i & \le 0 \\
 & & & & & & & & & X,Q & \le 0
\end{array}
$$

Readers may verify that at the optimal solution the Kuhn-Tucker conditions equate the price of wheat in the supply and demand markets as well as the quantity forming an overall equilibrium. The solution of this example arises from the file MARKETS and is given in Table 13.5. Now the question is, "What does the objective function represent?"

$$(0.75 - \tfrac{1}{2}*.0004X_b)X_b$$

The term is the area under the price curve for bread. Similarly, the other expressions are the integrals under the other curves. Thus, we have the integrals under the demand curves less the integrals under the supply curves leading us to a measure of the areas between the curves. The area between demand and supply functions is a measure of producers' plus consumers' surplus. Alternatively, this may be viewed as a technical behavioral objective whose purpose is to equate prices in markets.

This example again illustrates how price endogenous models can be constructed to account for multiple markets. Again, the nonlinear part of the model takes into account the price responsiveness in the demand and supply curves. This model has an explicit supply curve for the product wheat, composed of the aggregate of the two supply curves, as well as a demand curve which is the aggregate of demand for wheat in the production of three products.

13.4 Implicit Supply - Multiple Factors/Products

The above models involve explicit supply curves and production using a single input. However, one can depict multiple products, factors and production processes. Such models have exogenous factor supply and product demand curves, but implicit factor demand and product supply. A model of such a case is

$$\begin{array}{llllll}
\text{Max} & \sum_h \int_0^{Z_h} P_{dh}(Z_h)\, dZ_h & - \sum_i \int_0^{X_i} P_{si}(X_i)\, dX_i & & & \\
\text{s.t.} & Z_h & & - \sum_\beta \sum_k C_{h\beta k} Q_{\beta k} & \leq 0 & \text{for all h} \\
 & & - X_i & + \sum_\beta \sum_k a_{i\beta k} Q_{\beta k} & \leq 0 & \text{for all i} \\
 & & & \sum_k b_{j\beta k} Q_{\beta k} & \leq Y_{j\beta} & \text{for all j and } \beta \\
 & Z_h, & X_i, & Q_{\beta k} & \geq 0 & \text{for all i, h, k and } \beta
\end{array}$$

This problem assumes that a number of different types of firms (β) are being modeled. Each firm has a finite set of production processes (k) which depict particular ways of combining fixed factors (j) with purchased factors (i) to produce commodities (h). The symbols in the formulation are: $P_{dh}(Z_h)$ is the inverse demand function for the h^{th} commodity; Z_h is the quantity of commodity h that is consumed; $P_{si}(X_i)$ is the inverse supply curve for the i^{th} purchased input; X_i is the quantity of the i^{th} factor supplied; $Q_{\beta k}$ is the level of production process k undertaken by firm β; $C_{h\beta k}$ is the yield of output h from production process k; $b_{j\beta k}$ is the quantity of the j^{th} owned fixed factor used in producing $Q_{\beta k}$; $a_{i\beta k}$ is the amount of the i^{th} purchased factor used in producing $Q_{\beta k}$ and $Y_{j\beta}$ is the endowment of the j^{th} owned factor available to firm β.

An investigation of the Kuhn-Tucker conditions would show that the shadow price on the first and second rows are respectively the demand and supply prices. The conditions for the Q variable indicates that production levels are set so the marginal value of the commodities produced is less than or equal to the marginal costs of the owned and fixed factors for each $Q_{\beta k}$.

The model formulation assumes that: 1) the supply and demand equations are integrable (we will return to this assumption later, but for now we assume path independent integrals); and, 2) product demand and factor supply functions are truly exogenous to the model (i.e., there is no income effect).

The integral of the product demand and factor supply functions makes the objective function equal consumers' plus producers' surplus or net social benefit. The solution of the model generates equilibrium price and quantity for each output, and purchased input, along with the imputed values for the owned factors of production.

The model formulation assumes that the sector is composed of many micro-units, none of which can individually influence output or factor prices. Each micro-unit supplies output at the point where marginal cost equals product price, and utilizes purchased inputs at the point where the marginal value product of each purchased input equals its market price. Thus, the sectoral supply of output schedule corresponds to an aggregate marginal cost schedule, and the sectoral derived demand schedule for purchased inputs corresponds to the aggregate marginal value product schedule. Hence, the model does not take product supply or factor demand schedules as input, rather these schedules are derived internally based upon production possibilities, output demand and purchased input supply.

The competitive behavior simulating properties of this formulation provide a powerful tool for policy makers. Excepting centrally planned economies, the government cannot dictate production patterns consistent with its objectives. This formulation recognizes the difference and possible conflict between government and producer objectives (see Candler, Fortuny, and McCarl for elaboration). The model allows policy analysts to specify changes designed to meet some government objective, then simulate sectoral response to the policy change. The model does not assume participants respond to government "wants"; each producer optimally adjusts so as to maximize profits. Producer adjustment is endogenous to the model.

13.4.1 Example

Suppose we make some modifications to the block diagonal problem in Chapter 7 adding product demand and labor supply curves. Namely let us simplify the problem by only allowing sales from the first plant dropping the sales activities from the other plants. We will also specify linear product demand and labor supply curves. The curves are passed through a known price quantity point which has a particular elasticity at that point. Namely given the elasticity (□), and known price quantity point (P, Q) then the slope (b) is found as follows. We know that slope equals

$$b = \frac{\Delta P}{\Delta Q}$$

while the elasticity is

$$\in = \frac{\frac{\Delta Q}{\Delta P}}{\frac{Q}{P}}.$$

This implies that

$$b = \frac{\Delta P}{\Delta Q} = \frac{P}{\in Q}$$

and then if

$$P = a + bQ$$

the intercept is

$$a = P - bQ.$$

In setting up the model, the assumed price, quantity and elasticity by commodity as well as the computed intercept and slope are given below

Product Sale

Commodity	Price	Quantity	Elasticity	Computed Intercept (a)	Computed Slope (b)
Functional Chairs	82	20	-0.5	247	-8.2
Functional Tables	200	10	-0.3	867	-66.7
Functional Sets	600	30	-0.2	3600	-100
Fancy Chairs	105	5	-0.6	280	-35
Fancy Tables	300	10	-1.2	550	-25
Fancy Sets	1100	20	-0.8	2475	-68.8

Labor Supply

Plant	Price	Quantity	Elasticity	Intercept (a)	Slope (b)
Plant1	20	175	1	0	.114
Plant2	20	125	1	0	.160
Plant3	20	210	1	0	.095

The resultant model is given in Table 13.6 where the objective function terms for the demand variables marked with "w" equal

$$a * Q + 1/2\ b * Q^2$$

where Q depicts the quantity of the variable. The intercept and slope are as in the above table. Similarly, those supply terms marked with "-Z"", equal

$$-(c* Q + 1/2 \quad d\ \square\square Q^2)$$

where the intercept and slope are from the labor supply table above.

The solution to this problem is given in Table 13.7 (see file ACTANAL).

Note the balances give the market prices of chairs and tables while the plant level labor balances give the labor prices. The overall objective function value again equals consumers' plus producers' surplus.

13.5 Aggregation

An important sector modeling topic involves aggregation. Namely, the implicit supply model assumes that there are submodels present for each firm in the sector. This is usually not practical. Such models typically deal with the aggregate response across groups of firms. Two approaches have been proposed for the formation of such an aggregate representation. The first involves derivation of conditions under which a set of models can each represent more than one entity. Such conditions require that the problems have identical constraint matrices, proportional right hand sides and objective functions (Day, 1969). The second approach involves a reformulation of the programming model. We will deal further with this reformulation here.

The reformulation approach is based upon Dantzig and Wolfe decomposition and suggestions in McCarl. Dantzig and Wolfe based their scheme on the property that the solution to a subproblem or group of subproblems will occur at the extreme points of the subproblem(s). Thus, one can reformulate the problem so that it contains the extreme point solutions from the subproblems. Formally this can be expressed as follows. Given the problem

$$
\begin{array}{llllll}
\text{Max} & \sum_h \int_0^{Z_h} P_{dh}(Z_h)\, dZ_h & - \sum_i \int_0^{X_i} P_{si}(X_i)\, dX_i & & & \\
\text{s.t.} & Z_h & & - \sum_\beta \sum_k C_{h\beta k} Q_{\beta k} & \leq 0 & \text{for all h} \\
& - & X_i & + \sum_\beta \sum_k a_{i\beta k} Q_{\beta k} & \leq 0 & \text{for all i} \\
& & & \sum_k b_{j\beta k} Q_{\beta k} & \leq Y_{j\beta} & \text{for all j and } \beta \\
& Z_h, & X_i, & Q_{\beta k} & \geq 0 & \text{for all i, h, k and } \beta
\end{array}
$$

suppose we group the firms into subsets r_m where r_m depicts the m^{th} aggregate firm grouping. In turn, suppose we have a set of s feasible solutions $Q_{\square k}$ and add up their aggregate levels of production and input usage such that

$$Z_h^{ms} = \sum_{\beta \in r_m(\beta)} \sum_k C_{h\beta k} Q_{\beta k}^s \quad \textit{for all } m, h, \textit{and } s$$

$$X_i^{ms} = \sum_{\beta \in r_m(\beta)} \sum_k a_{i\beta k} Q_{\beta k}^s \quad \textit{for all } m, i, \textit{and } s$$

This in turn can be used in the aggregate problem:

$$
\begin{array}{llllll}
\text{Max} & \sum_h \int_0^{Z_h} P_{dh}(Z_h)\, dZ_h & - \sum_i \int_0^{X_i} P_{si}(X_i)\, dX_i & & & \\
\text{s.t.} & Z_h & & - \sum_m \sum_s Z_h \lambda_{ms} & \leq 0 & \text{for all h} \\
& - & X_i & + \sum_m \sum_s X_i^{ms} \lambda_{ms} & \leq 0 & \text{for all i} \\
& & & \sum_s \lambda_{ms} & \leq 1 & \text{for all m} \\
& Z_h, & X_i, & \lambda_{ms} & \geq 0 & \text{for all i, h, m and s}
\end{array}
$$

This model differs in two major ways from those above. First, the firm response variables have data requirements not in terms of individual production possibilities, but rather in terms of total production and consumption of the sector wide outputs and inputs accumulated across the firms in each group. In addition, rather than using individual resource constraints we now require a convex combination of the total output/input vectors. This will be feasible in the subproblems since any combination of two feasible subproblem solutions is feasible. Implicitly these solutions contain all the firm level resource restrictions and production possibilities coded within them.

The candidate solution vectors (i.e., the values of X_i^{ms},Z_h^{ms}) must be developed. These can be generated either by formally solving the linear programming subproblems for different prices or by selecting a historical set of observed feasible mixes or firms. This is discussed further in Onal and McCarl (1989, 1991).

13.5.1 Example

Suppose we have a problem with four production subproblems falling into two states where the first two firms are in state 1 and the second two are in state 2. Further suppose the firms each produce two goods and use miscellaneous inputs, labor and land. Suppose the land constraint is firm specific, the labor constraint is state specific and the miscellaneous inputs constraint is national. Suppose the supply and demand curves are in Table 13.8 and the rest of the data are as given in the tableau (Table 13.9). Aggregation is introduced into this problem by considering using two state level models. Suppose over time we have observed state crop mixes as in Table 13.10. We may then reformulate the model and, rather than include all the firms and resource constraints, we simply put in the total input and output use for the observed solutions (Table 13.11). The resultant national solutions before (see file BEFORAGG) and after (see file AFTERAGG) the aggregation process are given in Table 13.12a and 13.12b. Notice that there is not a great deal of difference in these optimum solutions.

This example is indicative of a general approach to such problems. Namely, if we were trying to represent all of the farms in a sector and could obtain production and input usages by state, we could modify the model to force a convex combination of historically observed activity. This is done in the sector models used by McCarl (1982b); Hamilton, McCarl and Adams; and Chang et al.

13.6 A Digression on the Assumptions

To formulate the above models or any other multi-product or multi-input model, one must assume integrability of product demand and purchased input supply functions as well as partial equilibrium. In this section, we will discuss these assumptions and suggest ways of relaxing them. Integrability requires that the Jacobians of the product demand equations and purchased input supply functions be symmetric (Hurwicz and Uzawa). The system of product demand functions is

$$P = G - HZ$$

and the system of purchased input supply functions is

$$R = E + FX$$

The Jacobians of the demand and supply equations are H and F, respectively. Symmetry of H and F implies that cross price effects across all commodity pairs are equal; i.e.,

$$\partial P_{dr} / \partial Q_{dh} = \partial P_{dh} / \partial Q_{dr} \quad \text{for all } r \neq h$$
$$\partial P_{sr} / \partial Q_{sh} = \partial P_{sh} / \partial Q_{sr} \quad \text{for all } r \neq h$$

In the case of supply functions, classical production theory assumptions yield the symmetry conditions. The Slutsky decomposition reveals that for the demand functions, the cross price derivatives consist of a symmetric substitution effect and an income effect. The integrability assumption requires the income effect to be identical across all pairs of commodities or to be zero.

Some authors argue that there need be no concern regarding symmetry. Since the objective function is a quadratic form. Then, given any square matrix, H, a quadratic form is the scaler quantity that results when H is pre- and post- multiplied by a conformable vector,

$$v = x'Hx$$

where v is the value of the quadratic form. Mathematically, if we replace H with the symmetric matrix B

$$B = 1/2 \, (H + H')$$

One can easily show that

$$X' BX = X' HX$$

Thus, if H is not symmetric, it can be replaced by B, and the value of the objective function remains unchanged. But, when the first order conditions are formed, the derivatives are altered. In particular if one integrates the above demand curve, we get

$$\frac{\partial (GZ - 1/2Z'HZ)}{\partial Z} = \frac{\partial GZ}{\partial Z} + \frac{\partial 1/2Z'HZ}{\partial Z} = G + 1/2Z'(H + H')$$

which would not give the demand price. Thus marginal cost and product price are no longer equilibrated.

Models can be formed which can handle asymmetry. Price and quantity variables can be included in the primal model (Plessner and Heady). Thus, both price and quantity equilibrium conditions are imposed on the primal problem, as contrasted with the above specification in which only quantity equilibrium conditions are imposed on the primal, and price equilibrium conditions are found in the dual. Another approach is linear complementarity programming (Takayama and Judge; Stoecker; or Polito). In this case, the objective function no longer represents consumers' plus producers' surplus. For further discussion, see Takayama and Judge or Martin.

The partial equilibrium assumption arises because the formulation does not incorporate the income generated by the sector as a simultaneous shifter of demand for products included in the model. If the entity modeled is small relative to the entire economy, this should not be a problem. If a major proportion of consumers included in the model are also producers, then the model inadequately describes the linkages in the economy. A formulation which does not require the partial equilibrium assumption was developed by Yaron, who specified a lagged relationship in which aggregate consumer demand in the current period is a function of income in the previous period. Norton and Scandizzo have relaxed this assumption in a simultaneous fashion in which demand is specified as a function of current consumer income. Integrability is a consequence as an income shifter is explicitly introduced, leaving only the symmetric substitution terms.

For further discussion of empirical specification of price endogenous models, see the review papers by McCarl and Spreen or Norton and Schiefer.

13.7 Imperfect Competition

So far, we have basically dealt with price endogeneity starting from Samuelson's approach, casting a set of first-order conditions and discovering the QP that would yield such a set. Another approach, however, can be taken. Suppose one begins with a classic LP problem involving two goods and a single constraint; i.e.,

$$
\begin{array}{llcll}
\text{Max} & P_1X & - & P_2Q & \\
\text{s.t.} & X & - & Q & \leq 0 \\
 & X, & & Q & \geq 0
\end{array}
$$

However, rather than P_1 and P_2 being fixed, suppose that we assume that they are functionally dependent upon quantity as given by

$$
\begin{array}{lcccc}
P_1 & = & a & - & bX \\
P_2 & = & c & + & dQ
\end{array}
$$

Now suppose one simply substitutes for P_1 and P_2 in the objective function. This yields the problem

$$
\begin{array}{llcccl}
\text{Max} & aX - bX^2 & - & cQ - dQ^2 & & \\
\text{s.t.} & X & & - Q & & \leq 0, \\
 & X & & Q & & \geq 0
\end{array}
$$

Note the absence of the 1/2's in the objective function. If one applies Kuhn-Tucker conditions to this problem, the conditions on the X variables, assuming they take on non-zero levels, are

$$
\begin{array}{lcccc}
a - 2bX & - & \lambda & = & 0 \\
-(c + 2dQ) & + & \lambda & = & 0
\end{array}
$$

The solution to this set of equations implies that the dual variable (□) is equated to something with twice the slope of the demand curve. Readers familiar with the imperfect competition literature will recognize this as an equation of marginal revenue with marginal cost. Such actions are only consistent with the behavior of perfectly discriminating monopolists - monopsonists. This indicates a couple of things about the approach to price equilibrating models: if one is not careful and does not put the integrals in, one simulates imperfect competition. In fact, there are four cases involving the integrals (1/2's in the quadratic case). Given the supply and demand relationships, one may model as follows

$[I]$ *Monopolist – Monopsonist*

$$\begin{aligned} Max \quad & X(a - bX) - Q(C + dQ) \\ & X - Q \leq 0 \end{aligned}$$

$[II]$ *Monopolist – Supply Competitor*

$$\begin{aligned} Max \quad & X(a - bX) - Q(C + 1/2dQ) \\ & X - Q \leq 0 \end{aligned}$$

$[III]$ *Demand Competitior – Monopsonist*

$$\begin{aligned} Max \quad & X(a - 1/2bX) - Q(C + dQ) \\ & X - Q \leq 0 \end{aligned}$$

$[IV]$ *Competitor in Both Markets*

$$\begin{aligned} Max \quad & X(a - 1/2bX) - Q(C + 1/2dQ) \\ & X - Q \leq 0 \end{aligned}$$

The solutions to these problems are graphed in Figure 13.3. Using the wheat problem, the numerical solutions shown in Table 13.13 are determined under the four alternative behavioral assumptions. This shows that one can obtain alternative forms of competition by selectively omitting or including integrals.

Nelson and McCarl provide a more general discussion of the topic of imperfect competition under the quadratic case. They show that in each of the demand and supply curves, if the term

$$\frac{n + 1}{2n}$$

is substituted for the 1/2, then one obtains a simulation of the effect of n firms discriminating against the demand or supply curves to this parameter is supplied. This particular term reduces to 1/2 when n approaches ∞, and 1 when n=1. Thus, it covers both the monopolistic and perfectly competitive cases. But also, for example, when n=2, the equation says to use a 3/4 to reflect two firms acting under imperfect competition against a particular supply curve. Readers should be careful in using this formulation, as it indicates how one discriminates against the entity which the particular supply or demand curve depicts, not how that entity discriminates against others. Nelson and McCarl present a more careful discussion on handling other forms of imperfect competition.

13.8 Conclusion

In the preceding sections, price endogenous models have been developed for spatial equilibrium, multi-market, multi-product, multi-factor models, aggregate, and imperfect competition. It should be clear that these models may be combined with our earlier formulations. For example, Spreen et al. integrated a multi-product industry formulation with a disequilibrium known life type formulation in a study of the livestock sector in Guyana.

These types of models have been used in many studies, as listed in the review book by Judge and Takayama, the review papers by McCarl and Spreen, Martin, and Norton and Schiefer.

13.9 References

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Table 13.1. GAMS Formulation of Simple Price Endogenous Example

```
OPTION LIMCOL = 0;
OPTION LIMROW = 0;

SETS      CURVEPARM  CURVE PARAMETERS  /INTERCEPT,SLOPE/
          CURVES     TYPES OF CURVES   /DEMAND,SUPPLY/

TABLE     DATA(CURVES,CURVEPARM) SUPPLY DEMAND DATA

                INTERCEPT    SLOPE
 DEMAND             6        -0.30
 SUPPLY             1         0.20

PARAMETERS  SIGN(CURVES)  SIGN ON CURVES IN OBJECTIVE FUNCTION
                   /SUPPLY -1,  DEMAND 1/

POSITIVE VARIABLES    QUANTITY(CURVES) ACTIVITY LEVEL

VARIABLES           OBJ              NUMBER TO BE MAXIMIZED

EQUATIONS           OBJJ             OBJECTIVE FUNCTION
                    BALANCE          COMMODITY BALANCE;

OBJJ..   OBJ =E= SUM(CURVES, SIGN(CURVES)*
                 (DATA(CURVES,"INTERCEPT")*QUANTITY(CURVES)
                 +0.5*DATA(CURVES,"SLOPE")*QUANTITY(CURVES)**2)) ;

BALANCE..    SUM(CURVES, SIGN(CURVES)*QUANTITY(CURVES)) =L= 0 ;

MODEL PRICEEND /ALL/ ;

SOLVE PRICEEND USING NLP MAXIMIZING OBJ ;
```

Table 13.2. Solution to Simple Price Endogenous Model

Variables	Level	Reduced Cost	Equation	Slack	Shadow Price
Q_d	10	0	Objective function	0	-1
Q_s	10	0	Commodity Balance	0	3

Table 13.3. Solution to Spatial Equilibrium Model

Objective function = 9193.6

Variables	Value	Reduced Cost	Equation	Level	Shadow Price
Supply			Supply Balance		
U.S.	79.6	0	U.S.	0	104.6
Europe	68.6	0	Europe	0	103.6
Demand			Demand Balance		
U.S.	45.4	0	U.S.	0	104.6
Europe	51.4	0	Europe	0	103.6
Japan	51.4	0	Japan	0	108.6
Shipments					
U.S. to U.S.	45.4	0			
U.S. to Europe	0	-4			
U.S. to Japan	34.2	0			
Europe to U.S.	0	-2			
Europe to Europe	51.4	0			
Europe to Japan	17.2	0			

Table 13.4. Solutions to Alternative Configurations of Spatial Equilibrium Model

	Undistorted	No Trade	Scenario Quota	Tax/Subsidy
Objective	9193.6	7506.3	8761.6	9178.6
U.S. Demand	45.4	62.5	61.5	46.4
U.S. Supply	79.6	62.5	63.5	78.6
U.S. Price	104.6	87.5	88.5	103.6
Europe Demand	51.4	60	40.7	50.4
Europe Supply	68.6	60	79.3	69.6
Europe Price	103.6	95	114.3	104.6
Japan Demand	51.4	0	40.7	51.4
Japan Price	108.6	160	119.3	108.6

Table 13.5. Solution to the Wheat Multiple Market Example

X_b	255.44
X_c	867.15
X_e	1608.72
Q_d	413.04
Q_i	1391.29
P_{db}	0.648
P_{dc}	0.540
P_{de}	3.239
P_{sd}	3.239
P_{si}	3.239
Shadow Price	3.239

Table 13.6. Implicit supply example

		Plant 1									Plant 2									Plant 3													
		Make Sets		Make Table		Sell Table		Sell Chair		Hire	Transport Chair		Make Fine Chair			Make Fancy Chair			Hire	Transport Table		Transport Chair		Make Table		Make Fine Chair			MakeFancyChair			Hire	
		Fn	Fc	Fn	Fc	Fn	Fc	Fn	Fc	Lab	Fn	Fc	Nm	MxS	MxL	Nm	MxS	MxL	Lab	Fn	Fc	Fn	Fc	Fn	Fc	Nm	MxS	MxL	Nm	MxS	MxL	Lab	
	Objective	w	w	-80	-100	w	w	w	w	-z	-5	-5	-15	-16	-16	-25	-26	-26	-z	-5	-5	-5	-5	-80	-100	-15	-16	-16	-25	-26	-26	-z	
PLANT 1	Tables Fine	1		-1		1																											≤ 0
	Tables Fancy		1		-1		1																										≤ 0
	Chair Fine	4						1			-1									-1													≤ 0
	Chair Fancy		6						1			-1									-1												≤ 0
	Labor			3	5																	-1											≤ 0
	Top Capacity			1	1					-1													-1										≤ 50
PLANT 2	Chair Fine										1		-1	-1	-1																		≤ 0
	Chair Fancy											1				-1	-1	-1															≤ 0
	Small Lathe												0.8	1.3	0.2	1.2	1.7	0.5															≤140
	Large Lathe												0.5	0.2	1.3	0.7	0.3	1.5															≤ 90
	Carver												0.4	0.4	0.4	1	1	1															≤ 120
	Labor												1	1.05	1.1	0.8	0.82	0.84	-1														≤ 0
PLANT 3	Tables Fine																			1				-1									≤ 0
	Tables Fancy																				1				-1								≤ 0
	Chair Fine																					1				-1	-1	-1					≤ 0
	Chair Fancy																						1						-1	-1	-1		≤ 0
	Small Lathe																									0.8	1.3	0.2	1.2	1.7	0.5		≤130
	Large Lathe																									0.5	0.2	1.3	0.7	0.3	1.5		≤100
	Carver																									0.4	0.4	0.4	1	1	1		≤110
	Labor																							3	5	1	1.05	1.1	0.8	0.82	0.84	-1	≤ 0
	Top Capacity																							1	1								≤ 40

Table 13.7. Solution of the Implicit Supply Example

Rows			Slack	Shadow Price
Objective			95779.1	
PLANT 1	Table	FC	0	165.1
	Table	FY	0	228.5
	Chair	FC	0	85.6
	Chair	FY	0	110.8
	Labor		0	21.7
	Top Capacity		0	20.0
PLANT 2	Chair	FC	0	80.6
	Inventory	FY	0	105.8
	Small Lathe		0	35.6
	Large Lathe		0	28.0
	Carver		29.02	0
	Labor		0	23.1
PLANT 3	Table	FC	0	145.1
	Inventory	FY	0	208.5
	Chair	FC	0	78.6
	Inventory	FY	0	103.8
	Small Lathe		0	35.1
	Large Lathe		0	27.6
	Carver		0.80	0
	Labor		0	21.7
	Top Capacity		12.69	0

	Variable Names		Level	Reduced Cost
PLANT 1	Sell FC Set			0
	Sell FY Set		23.0	0
	Sell FC Tables		10.5	0
	Sell FY Tables		12.9	0
	Sell FC Chairs		19.6	0
	Sell FY Chairs		4.8	0
	Make Table FC		30.1	0
	Make Table FY		20.0	0
	Hire Labor		189.9	0
PLANT 2	Transport FC Chair		105.0	0
	Transport FY Chair		48.9	0
	Make Table FC		0	-69.3
	Make Table FY		0	-115.5
		N	105.1	0
	Make FC Chair	S	0	-11.6
		L	0	-4.8
		N	44.9	0
	Make FY Chair	S	0	-7.76
		L	4.0	0
	Hire Labor		144.4	0
PLANT 3	Transport FC Table		11.4	0
	Transport FY Table		15.9	0
	Transport FC Chair		38.2	0
	Transport FY Chair		93.9	0
	Make FC Table		11.4	0
	Make FY Table		15.9	0
	Make FC Chair	N	38.2	0
		S	0	-11.3
		L	0	-4.7
	Make FY Chair	N	75.0	0
		S	0	-7.6
		L	19.0	0
	Hire Labor		227.9	0

Table 13.8. Demand and Supply Parameters for Aggregation Example

	Price	Quantity	Elasticity
Product Demands			
Cotton	225	3326	-1.5
Corn	2.10	1087	-1.1
Hired Labor Supply			
State 1	5	78.7	0.5
State 2	4.5	68.1	1.2

Table 13.9. Before Aggregation Formulation of Aggregation Example

	Sales		Hired Labor		Farm Hired Labor				Produce Farm 1		Produce Farm 2		Produce Farm 3		Produce Farm 4		Misc Inputs		
	corn	cotton	State 1	State 2	Farm 1	Farm 2	Farm 3	Farm 4	corn	cotton	corn	cotton	corn	cotton	corn	cotton			
Obj. Func.	a	a	-b	-b													-1		
Misc Inputs									80	303	95	278	110	437	70	300	-1	=	0
Labor State 1			-1		1	1												≤	0
Labor State 2				-1			1	1										≤	0
Farm Labor Farm 1					-1				10.4	14.5								≤	15.5
Farm Labor Farm 2						-1					12.9	17.5						≤	13.1
Farm Labor Farm 3							-1						12.2	24.5				≤	11.5
Farm Labor Farm 4								-1							9.6	14		≤	11.3
Product Corn	1								-120		-180		-150		-150			≤	0
Balance Cotton		1								-2.2		-2.6		-3.1		-2.5		≤	0
Land Available Farm 1									1	1								≤	6
Land Available Farm 2											1	1						≤	4
Land Available Farm 3													1	1				≤	5
Land Available Farm 4															1	1		≤	3

Table 13.10. Crop Mix Data for use in Aggregation Example

		Mix 1		Mix 2	
Region	Farm	Corn	Cotton	Corn	Cotton
State 1	Farm1	.3	.7	.5	.5
	Farm 2	.1	.9	.3	.7
State 2	Farm1	.6	.4	.75	.25
	Farm 2	.55	.45	.6	.4

Table 13.11. Aggregation Example after Aggregation

	Sales		Hired Labor		Crop Mixes				Misc		
					State1		State 2				
	corn	cotton	State 1	State 2	Mix 1	Mix 2	Mix 1	Mix 2	Inputs		
Obj. Func.	a	a	-b	-b					-1		
Misc Inputs					2455	2041	1725	1445	-1	=	0
Labor State 1			-1		119.2	110.6				≤	0
Labor State 2				-1			97.6	87.7		≤	0
Product Corn	1				-288	-576	-9.6	-6.9		≤	0
Balance Cotton		1			-18.6	-13.9	-698	-833		≤	0
Convexity State1					1	1				≤	1
Convexity State2							1	1		≤	1

Table 13.12. Solutions of Aggregation Example

A Before Aggregation

Rows	Slack	Shadow Price	Variable	Level	Reduced Cost
Objt	7777.4				
Misc Inputs	0	-1.000	Sales Cotton	31.6	0
			Sales Corn	967.4	0
State Labor State 1	0	2.318			
State Labor State 2	0	4.288	Hired Labor State 1	57.6	0
			Hired Labor State 2	64.3	0
Farm Labor 1	0	1.16			
Farm Labor 2	0	1.16	Hire Labor Farm 1. State 1	35.8	0
Farm Labor 3	0	2.14	Hire Labor Farm 2. State 2	21.8	0
Farm Labor 4	0	2.14	Hire Labor Farm 3. State 3	55.5	0
			Hire Labor Farm 4. State 4	8.8	0
Cotton	0	232.6			
Corn	0	2.31	Corn. Farm 1	0	-6.2
			Cotton. Farm 1	6.0	0
Land Farm 1	0	191.4	Corn. Farm 2	2.9	0
Land Farm 2	0	305.9	Cotton. Farm 2	1.1	0
Land Farm 3	0	230.8	Corn. Farm 3	0	-20.4
Land Farm 4	0	255.9	Cotton. Farm 3	5.0	0
			Corn. Farm 4	3.0	0
			Cotton. Farm 4	0	-5.0
			Misc Inputs	4799	0

B After Aggregation

Rows	Slack	Shadow Price	Variable	Level	Reduced Cost
Obj. Func.	7052.2		Sales Corn	28.2	0
Cost	0	1	Sales Cotton	985.5	0
Labor State 1	0	10.1	Hire Labor State 1	119.2	0
Labor State 2	0	6.1	Hire Labor State 2	97.5	0
Product Bal. Corn	0	247.9	Crop State 1 Mix 1	1	0
Product Bal. Cotton	0	2.28	Crop State 1 Mix 2	0	-12.9
Convex State 1	1	1603.4	Crop State 2 Mix 1	1	0
Convex State 2	1	1641.5	Crop State 2 Mix 2	0	-21.6
			Cost	4180	0

Table 13.13. Alternative Solutions to Wheat Multiple Markets Example under Varying Competitive Assumptions

	I	II	III	IV
X_b	127.718	142.335	226.067	255.46
X_c	433.579	449.821	834.526	867.16
X_e	804.357	1096.705	1021.340	1608.71
Q_d	206.521	393.533	216.311	413.04
Q_i	695.642	806.589	989.330	1391.29
P_{db}	0.699	0.693	0.660	0.649
P_{dc}	0.669	0.665	0.550	0.540
P_{de}	3.3196	3.29033	3.2979	3.239
P_{sd}	2.6196	3.18066	2.6489	3.239
P_{si}	3.6196	3.18066	3.1999	3.239
Shadow Price	3.2391	3.18066	3.2979	3.239