1. An introduction to dynamic optimization -- 
Optimal Control and Dynamic Programming
AGEC 642 - 2018

I. Overview of optimization
Optimization is a unifying paradigm in almost all economic analysis. So before we start, let’s think about optimization. The tree below provides a very nice general representation of the range of optimization problems that you might encounter. There are two things to take from this. First, all optimization problems have a great deal in common: an objective function, constraints, and choice variables. Second, there are lots of different types of optimization problems and how you solve them will depend on the branch on which you find yourself.

In terms of the entire tree of all optimization problems, the ones that could be solved analytically would represent a couple of leaves at best – numerical methods must be used to solve the rest. Fortunately, a great deal can be learned about economics by studying those problems that can be solved analytically.

![Optimization Tree](https://example.com/optimization_tree)

**Source:** The Optimization Technology Center: [http://www.ece.northwestern.edu/OTC/](http://www.ece.northwestern.edu/OTC/)

In this course we will use both analytical and numerical methods to solve dynamic optimization problems, problems that have two common features: the objective function is a linear aggregation over time, and a set of variables called the state variables are constrained across time. And so we begin …
II. Introduction – A simple 2-period consumption model

Consider the simple consumer's optimization problem:

\[
\max_z u(z_a, z_b) \quad \text{s.t.} \quad p_a z_a + p_b z_b \leq x
\]

[pay attention to the notation: \(z\) is the vector of choice variables and \(x\) is the consumer's exogenously determined income.]

Solving the one-period problem should be familiar to you. What happens if the consumer lives for two periods, but has to survive off of the income endowment provided at the beginning of the first period? That is, what happens if her problem is

\[
\max_{z_2} U(z_{1a}, z_{1b}, z_2) = U(z_1, z_2) \quad \text{s.t.} \quad p_1 z_1 + p_2 z_2 \leq x_1,
\]

where the constraint uses matrix notation with \(p = [p_a, p_b]\) refers to a price vector and \(z_i = [z_{ia}, z_{ib}]\). We now have a problem of dynamic optimization. When we chose \(z_i\), we must take into account how it will affect our choices in period 2.

We’re going to make a huge (though common) assumption and maintain that assumption throughout the course: utility is additively separable across time\(^1\):

\[
u(z) = u(z_1) + u(z_2).
\]

Clearly one way to solve this problem would be just as we would a standard static problem: set up a Lagrangian and solve for all optimal choices simultaneously. This may work here, when there are only 2 periods, but if we have 100 periods (or even an infinite number of periods) then this could get really messy. This course will develop methods to solve such problems.

This is a good point to introduce some very important terminology:

- All dynamic optimization problems have a time horizon. In the problem above \(t\) is discrete, \(t=\{1,2\}\), but \(t\) can also be continuous, taking on every value between \(t_0\) and \(T\), and we can solve problems where \(T \rightarrow \infty\).

- \(x_t\) is what we call a state variable because it is the state that the decision-maker faces in period \(t\). Note that \(x_t\) is parametric (i.e., it is taken as given) to the decision-maker’s problem in \(t\), and \(x_{t+1}\) is parametric to the choices in period \(t+1\). However, \(x_{t+1}\) is affected by the choices made in \(t\). The state variables in a problem are those that a decision maker takes as given when making his or her choices in each period.

- A state equation defines the intertemporal changes in a state variable. This equation is sometimes referred to as the equation of motion or the transition equation.

\(^1\) See Deaton and Muellbauer (137-142) on the negative implications of assuming preferences are additive
- $z_t$ is the vector of $t$th period choice variables. Choice variables determine the (expected) payoff in the current period and the (expected) state next period. These variables are also referred to as control or action variables and I will use all these terms interchangeably.

  *To distinguish state & control variables, I like to say, “You wake up in the morning, look at your state variables, make decisions about your control variables, then go back to sleep.”*

- $p_a$ and $p_b$ are parameters of the model. They are held constant or change exogenously and deterministically over time.

- Finally, we have what I call intermediate variables. These are variables that are really functions of the state and control variables and the parameters. For example, in the problem considered here, one-period utility might be carried as an intermediate variable. In firm problems, production or profit might be other intermediate variables while productivity or profitability (a firm’s capacity to generate output or profits) could be state variables. Do you see the difference? When you formulate a problem it is very important, but often difficult, to distinguish state variables from intermediate variables (see PS#1).

- The benefit function tells the instantaneous or single period net benefits that accrue to the planner during the planning horizon. In our problem $u(z_t)$ is the benefit function. Despite its name, the benefit function can be positive or negative. For example, a function that defines the cost in each period can be the benefit function.

- In many problems there are benefits (or costs) that accrue after the planning horizon. This is captured in models by including a salvage value, which is usually a function of the terminal stock. Since the salvage value occurs after the planning horizon, it cannot be a function of the control variables, though it can be a separate optimization problem in which choices are made.

- The sum (or integral) over the planning horizon plus the salvage value determines the objective function. We usually use discounting when we sum up over time. Pay close attention to this – the objective function is not the same as the benefit function.

- All of the problems that we will study in this course fall into the general category of Markov decision processes (MDP). In an MDP the probability distribution over the states in the next period is wholly determined by the current state and current actions. One important implication of limiting ourselves to MDPs is that, typically, history does not matter, i.e. $x_{t+1}$ depends on $z_t$ and $x_t$, irrespective of the value of $x_{t-1}$. When history is important in a problem, then the relevant historical variables must be explicitly included as state variables.

- A Formal Statement of the Optimization Problem is a set of mathematical expressions including the objective function and all the constraints. The constraints include the state equation, any conditions that must be satisfied at the beginning and end of the time horizon, and any constraints that restrict choices between the beginning and end. At a minimum, dynamic optimization problems must include the objective function, the state equation(s) and initial conditions for the state variables.
In sum, the problems that we will study will have the following features. In each period or moment in time the decision maker takes as given the state variables and parameters, then makes optimal choices for the control variables taking into account the objective function and state equations. The combination of \(x_t\) and \(z_t\) generates immediate benefits and costs and determines the probability distribution over \(x\) in the next period the rate of change in \(x_t\).

Instead of using brute force to find the solutions of all the \(z\)'s in one step, we reformulate the problem. Let \(x_1\) be the endowment which is available in period 1, and \(x_2\) be the endowment that remains in period 2. Following from the budget constraint, we can see that \(x_2 = x_1 - p'z_1\), with \(x_2 \geq 0\). In this problem \(x_2\) defines the state that the decision maker faces at the start of period 2. The equation which describes the change in the \(x\) from period 1 to period 2, \(x_2 - x_1 = -p'z_1\), is the state equation.

We now rewrite our consumer’s problem, this time making use of the state equation:

\[
\max_{z_t} \sum_{t=1}^{2} u_t(z_t) \quad \text{s.t.} \\
x_{t+1} - x_t = -p'z_t \\
x_{t+1} \geq 0 \quad \left\{ \begin{array}{l} t = 1, 2 \\ x_1 \text{ fixed} \end{array} \right.
\]

(1)

We now have a nasty little optimization problem with four constraints, two of them inequality constraints. Not fun. This course will help you solve and understand these kinds of problems. Note that this formulation is quite general in that you could easily write the n-period problem by simply replacing the 2's in (1) with \(n\).

III. The OC (optimal control) way of solving the problem

We will solve dynamic optimization problems using two related methods. The first of these is called optimal control. Optimal control makes use of Pontryagin’s maximum principle.

First note that for most specifications, economic intuition tells us that \(x_2 > 0\) and \(x_3 = 0\). Hence, for \(t = 1, 2\), we can suppress inequality constraint in (1). We’ll use the fact that \(x_3 = 0\) at the very end to solve the problem.

Write out the Lagrangian of (1):

\[
L = \sum_{t=1}^{2} \left[ u_t(z_t, x_t) + \lambda_t (x_t - x_{t+1} - p'z_t) \right]
\]

(2)

where we include \(x_t\) in \(u(\cdot)\) for completeness, though in this case \(\partial u/\partial x = 0\).
More terminology

In optimal control theory, the variable $\lambda_t$ is called the **costate variable** and, following the standard interpretation of Lagrange multipliers, at its optimal value $\lambda_t$ is equal to the marginal value of relaxing the constraint. In this case, that means that $\lambda_t$ is equal to the marginal value of the state variable, $x_t$. The costate variable plays a critical role in dynamic optimization.

The first order conditions (FOCs) for (2) are standard:

$$
\frac{\partial L}{\partial z_t} = \frac{\partial u}{\partial z_t} - \lambda_t p_i = 0, \quad i = a, b, \quad t = 1, 2
$$

$$
\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda_1 + \lambda_2 = 0
$$

[note that $x_1$ is not a choice variable since it is fixed at the outset and $x_3$ is equal to zero]

$$
\frac{\partial L}{\partial \lambda_i} = (x_t - x_{t+1} - p'z_t) = 0, \quad t=1,2.
$$

We now use a little notational change that simplifies this problem and adds some intuition (we'll see how the intuition arises in later lectures). That is, we define a function known as the **Hamiltonian** where

$$
H(z_t, x_t, \lambda_t) = u(z_t, x_t) + \lambda_t (-p'z_t).
$$

Some things to note about the Hamiltonian:

- the $t^{th}$ Hamiltonian only includes current variables: $z_t$, $x_t$ and $\lambda_t$,
- unlike in a Lagrangian, only the right-hand side of state equation appears after $\lambda_t$.

In the left column of table below we present the well-known FOCs of the Lagrangian. On the right we present the derivative of the Hamiltonian with respect to the same variables. Comparing the two sides, we can see what we would have to place on the right-hand side of the derivatives of the Hamiltonian to obtain the same optimum as when the Lagrangian is used.

[Fill in the blanks in the right column before proceeding]

<table>
<thead>
<tr>
<th>Lagrangian</th>
<th>Hamiltonian</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = \sum_{i=1}^2 [u_i(z_{i,t}, z_{i,b}) + \lambda_i(x_t - x_{t+1} - (p_a z_{i,t} + p_b z_{i,b})]</td>
<td>H = u(z_t, x_t) + \lambda_t (-p'z_t)$</td>
</tr>
<tr>
<td>Standard FOCs</td>
<td>$\partial H / \partial_-$</td>
</tr>
<tr>
<td>$\frac{\partial L}{\partial z_t} = \frac{\partial u}{\partial z_t} - \lambda_t p_i = 0, \quad t=1,2, \quad i=a,b$</td>
<td>$\frac{\partial H}{\partial z_t} = \frac{\partial u}{\partial z_t} - \lambda_t p = _____$</td>
</tr>
<tr>
<td>$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda_1 + \lambda_2 = 0$</td>
<td>$\frac{\partial H}{\partial x_2} = \frac{\partial u}{\partial x_2}(z_2, x_2) = _____$</td>
</tr>
<tr>
<td>$\frac{\partial L}{\partial \lambda_t} = x_t - x_{t+1} - p'z_t = 0, \quad t=1,2, \quad i=a,b$</td>
<td>$\frac{\partial H}{\partial \lambda_t} = -p'z_t = _____$</td>
</tr>
</tbody>
</table>
Hence, we see that for the solution using the Hamiltonian to yield the same maximum the following conditions must hold

1. \( \frac{\partial H}{\partial z_t} = 0 \) \implies \text{The Hamiltonian should be maximized w.r.t. the control variable at every point in time.}

2. \( \frac{\partial H}{\partial x_t} = \lambda_{t-1} - \lambda_t \) for \( t>1 \) \implies \text{The costate variable changes over time at a rate equal to minus the marginal value of the state variable to the Hamiltonian.}

3. \( \frac{\partial H}{\partial \lambda_t} = x_{t+1} - x_t \) \implies \text{The state equation must always be satisfied.}

When we combine these with a 4\textsuperscript{th} condition, called the transversality condition (how we transverse over to the world beyond \( t=1,2 \)) we're able to solve the problem. In this case the condition that \( x_3 = 0 \) (which for now we will assume to hold without proof) serves that purpose. We'll discuss the transversality condition in more detail in a few lectures.

These four conditions are the starting points for solving most optimal control problems and sometimes the FOCs alone are sufficient to understand the economics of a problem. However, if we want an explicit solution, then we would solve this system of equations.

In this class most of the OC problems we’ll face are in continuous time. The parallels between the discrete time case presented here and the continuous time case should be obvious when we get there.

IV. The DP (Dynamic Programming) way of solving the problem

The second way that we will solve dynamic optimization problems is using Dynamic Programming. DP is about \textbf{backward induction} – thinking backwards about problems. Let’s see how this is applied in the context of the 2-period consumer's problem.

Imagine that the decision-maker is now in period 2, having already used up part of her endowment in period 1, leaving \( x_2 \) to be spent. In period 2, her problem is simply

\[
V_2(x_2) = \max_{z_2} u_2(z_2) \quad \text{s.t.} \quad p'z_2 \leq x_2
\]

If we solve this problem, we can easily obtain the function \( V(x_2) \), which tells us the maximum utility that can be obtained if she arrives in period 2 with \( x_2 \) dollars remaining. The function \( V(\cdot) \) is equivalent to the indirect utility function with \( p_a \) and \( p_b \) suppressed. The period 1 problem can then be written

\[
\max_{z_1} u(z_1) + V_2(x_2) \quad \text{s.t.} \quad x_2 = x_1 - p'z_1.
\]

The value of having \( x_1 \) in period one is the solution to this problem, i.e.

\[
V_1(x_1) = \max_{z_1} u(z_1) + V_2(x_2).
\]

This equation is known as the Bellman’s equation and it is the cornerstone of dynamic programming.
Note that we have implicitly assumed an interior solution so that the constraint requiring that $x_3 \geq 0$ is assumed to hold with an equality and can be suppressed. Once we know the functional form of $V(\cdot)$, (3) becomes a simple static optimization problem and its solution is straightforward. If the functional form of $V(x_3)$ has been found, then we can write out Lagrangian of the first period problem,

$$L = u(z_1) + V'(x_2) + \lambda_1(x_1 - p'z_2 - x_2).$$

We see that the economic meaning of the costate variable, $\lambda_1$, is just as in the OC setup, i.e., it is equal to the marginal value of a unit of $x_1$.

A major challenge is that we do not have an explicit functional form for $V(\cdot)$ and as the problem becomes more complicated, obtaining a functional form becomes more difficult, even impossible for many problems. Hence, the trick to solving DP problems is to find the function $V(\cdot)$.

V. Are OC and DP equivalent? Yes.
As we will see throughout this course, either of these approaches can be used to solve a dynamic optimization problem. In this section we will quickly show that the first-order conditions for a simple problem are equivalent.

Consider the continuous-time dynamic optimization problem,

$$\max_{x_t} \int_{t=0}^{T} u(x_t, z_t) dt \quad \text{s.t.} \quad \dot{x}_t = f(x_t, z_t),$$

where, as we will discuss in Lecture 2, $\dot{x}_t = \partial x_t / \partial t$. The discrete-time analog of this problem is

$$\max_{x_j} \sum_{j=0}^{T/\Delta} u(x_j, z_j) \Delta \quad \text{s.t.} \quad x_{j+\Delta} = x_j + \Delta f(x_j, z_j),$$

where $\Delta$ is some fraction of a period. For example, if $\Delta = 0.5$, then there are two increments per period that go from $j=0$ to $j=2T=7/0.5$.

The Hamiltonian and the value function for these two problems are:

$$H(x_j, z_j, \lambda_j) = \Delta u(x_j, z_j) + \lambda_j \Delta f(x_j, z_j), \quad \text{and} \quad V(x_j, t) = \Delta u(x_j, z_j) + V(x_{j+\Delta}, t + \Delta) \quad \text{where} \quad x_{j+\Delta} = x_j + \Delta f(x_j, z_j).$$

First, we show the equivalence of the FOCs w.r.t the control variable as $\Delta \to 0$. The first order condition for the Hamiltonian is, as above,

$$\frac{\partial H}{\partial z_j} = \Delta u_z + \lambda_j \Delta f_z = 0.$$ 

For the value function, we know that at the optimum value for $z$, $\partial V / \partial z = 0$, i.e.
\[ \frac{\partial V}{\partial z_t} = \Delta u_z + \Delta \frac{\partial V(x_{t+\Delta}, t + \Delta)}{\partial x_{t+\Delta}} f_z = 0. \]

Dividing by \( \Delta \) and taking the limit as \( \Delta \to 0 \) so that \( t + \Delta \to t \), we have \( u_z + \frac{\partial V(x_t, t)}{\partial x_t} f_z = 0 \). Finally, since \( \frac{\partial V(x_t, t)}{\partial x_t} = \lambda_t \), it follows that

\[
\lim_{\Delta \to 0} \frac{\partial V}{\partial z_t} = \frac{\partial H}{\partial z_t} = 0. \quad \Box
\]

Next, we can show the equivalence of the FOC w.r.t the state variable, \( x_t \), as \( \Delta \to 0 \).

Again, we stated above that the FOC of the Hamiltonian for the state variable is

\[
\frac{\partial H}{\partial x_t} = \Delta \frac{\partial u}{\partial x_t} + \lambda_t \Delta \frac{\partial f}{\partial x_t} = \lambda_t - \lambda_{t+\Delta}. \quad (4)
\]

If we divide the middle and last part of this equality by \( \Delta \) and then take the limit as \( \Delta \to 0 \) this becomes

\[
\frac{\partial u}{\partial x_t} + \lambda_t \frac{\partial f}{\partial x_t} = -\frac{\partial \lambda_t}{\partial t}. \quad (5)
\]

For the Bellman’s equation, since \( x_t \) is not a choice variable – it is fixed at time \( t \) – the partial derivative is not set to zero; it is simply

\[
\frac{\partial V(x_t, t)}{\partial x_t} = \Delta \frac{\partial u}{\partial x_t} + \Delta \frac{\partial V(x_{t+\Delta}, t + \Delta)}{\partial x_{t+\Delta}} \frac{\partial x_{t+\Delta}}{\partial x_t}. \quad (6)
\]

Notice that there is some nice intuition in (6): the marginal value of the state variable is equal to the sum of what you get out of it in the first period of length \( \Delta \), \( \Delta u_x \), plus what you get in the future because you have more \( x_t \):

\[
\frac{\partial V(x_t, t)}{\partial x_t} = \Delta \frac{\partial u}{\partial x_t} + \lambda_{t+\Delta} \left(1 + f_x \Delta\right). \quad \text{Hence, (6) can be rewritten}
\]

\[
\lambda_t = \Delta u_x + \lambda_{t+\Delta} \left(1 + f_x \Delta\right), \quad \text{subtracting} \lambda_{t+\Delta} \text{from both sides and dividing by} \Delta \text{we obtain}
\]

\[
\frac{\lambda_t - \lambda_{t+\Delta}}{\Delta} = u_x + \lambda_{t+\Delta} f_x. \quad \text{Again we take the limit at} \Delta \to 0, \text{which in this case gives us}
\]

\[
-\frac{\partial \lambda_t}{\partial t} \text{ on the LHS, to obtain} -\frac{\partial \lambda_t}{\partial t} = u_x + \lambda_t f_x, \text{which is the same as FOC for the Hamiltonian, (5).}
\]

Finally, it is obvious that the state equation in both formulations must hold, regardless of the length of \( \Delta \). Hence, we have shown that the two approaches are equivalent.

VI. Summary

- OC problems are solved using the vehicle of the Hamiltonian, which must be maximized at each point in time.
- DP is about backward induction.
- Both techniques are equivalent to standard Lagrangian techniques and the interpretation of the shadow price, \( \lambda_t \), is the same.
VII. Reading for next lecture
Leonard and Van Long, chapter 2.

VIII. References