8. A more formal introduction to Dynamic Programming and Numerical DP
AGEC 642 - 2020

I. Some DP lingo
• The Bellman’s equation is an equation like: $V_t(x_t) = \max_{z_t} u(x_t, z_t) + V_{t+1}(x_{t+1})$
• We assume that the state variable $x_t \in X \subset \mathbb{R}^n$
• Bellman’s equation is a functional equation in that it maps from the function $V_{t+1} : X \rightarrow \mathbb{R}$ to the function $V_t : X \rightarrow \mathbb{R}$.
• Bellman’s equation is a recursive expression in that successive functions are determined by an operation on the preceding function, i.e. $V_T \rightarrow V_{T-1} \rightarrow V_{T-2} \rightarrow \ldots$

A. Important features of the DP framework
1. The Bellman’s equation at any point in time is a static optimization problem and everything that we know about solving such problems applies here.
2. If you look back at Dorfman’s framework (Lecture 5), you will find a lot of similarities between that structure and what we have here.
3. Hence, just as in optimal control, there will be a co-state variable $\lambda_t$ which is equal to the marginal value of an additional unit $x_t$, i.e. $\lambda_t = \partial V / \partial x_t$ or, if $x$ is discrete as in many of our examples, i.e. taken from the set $\{x_1, x_2, \ldots, x_n\}$, then $\lambda$ at a point $x_i$ could be calculated as $\lambda(x_i) = \frac{V(x_i) - V(x_{i-1})}{x_i - x_{i-1}}$.\footnote{Notice that here I define $\lambda(x_i)$ based on the one-sided difference in the negative direction, i.e. comparing $V(x_i)$ and $V(x_{i-1})$. We could have used a forward difference, i.e. $x_i$ relative to $x_{i+1}$, or an average of the two one-sided differences. There is no correct approach; but it is important to tell your reader what you did.}
4. The marginal future consequence of choices today is equal to
   $\frac{\partial V}{\partial x_{t+1}} \frac{\partial x_{t+1}}{\partial z_t} = \lambda(x_{t+1}) \frac{\partial x_{t+1}}{\partial z_t}$ or the analog for a discrete case.
5. If the problem were stochastic, then the Bellman’s equation would look like
   $V_t(x_t) = \max_{z_t} \left\{ u(x_t, z_t, \varepsilon_t) + V_{t+1} \left( g(x_t, z_t, \varepsilon_t) \right) \right\}$ where $\varepsilon$ is a random variable. Other than the addition of the expectation, however, we see that the underlying structure is basically unchanged. The fact that the transition from deterministic to stochastic problems is so straightforward is a big advantage of dynamic programming.
6. In AGEC 642 we will typically consider problems with a discrete-time specification. However, the Bellman’s equation can be analyzed in a continuous time framework. Continuous-time dynamic programming is discussed in detail in Stokey and Lucas (1989).

II. A general theoretical framework for the consideration of dynamic programming problems
The general problem that we will seek to solve through the use of dynamic programming is to maximize the expected benefits over our planning horizon. Benefits in period $t$ will
be written \( u(z_t, x_t, \varepsilon_t, t) \), where \( z_t \) represents vector of choices in \( t \), \( x_t \) a vector of state variables, and \( \varepsilon_t \) represents a vector of random variables. The value of your terminal stock, equivalent to the salvage value, will be written \( S(x_{T+1}) \).

Hence, the decision maker’s objective is to maximize

\[
E \left\{ u \left( z_t, x_t, \varepsilon_t, t \right) + u \left( z_{t+1}, x_{t+1}, \varepsilon_{t+1}, t+1 \right) + u \left( z_{t+2}, x_{t+2}, \varepsilon_{t+2}, t+2 \right) + \ldots + u \left( z_T, x_T, \varepsilon_T, T \right) + S \left( x_{T+1} \right) \right\}
\]

where \( E_t \) is the expectation operator conditional on the information available in \( t \).

The value function \( V(x_t) \) is defined as

1. \( V \left( x_t, t \right) \equiv \max_{z_t} E_t \left\{ \sum_{s=1}^T u \left( z_s, x_s, \varepsilon_s, s \right) + S \left( x_{T+1} \right) \right\} \).

If this objective is to be achieved, we must assume that the optimal choices will be made not only in \( t \) but in \( t+1, t+2, \ldots \). Hence,

\[
V \left( x_{t+1}, t+1 \right) \equiv \max_{z_t} E_{t+1} \left\{ \sum_{s=t+1}^T u \left( z_s, x_s, \varepsilon_s, s \right) + S \left( x_{T+1} \right) \right\}
\]

which can then be substituted into 1, to get

2. \( V \left( x_t, t \right) = \max_{z_t} E_t \left\{ u \left( z_t, x_t, \varepsilon_t, t \right) + V \left( x_{t+1}, t+1 \right) \right\} \)

with the terminal condition \( V \left( X_{T+1}, T + 1 \right) = S \left( X_{T+1} \right) \).

While \( V(x_t, t) \) is the max, we are probably also interested in the choice variables that actually solve this problem. The policy function, \( z^*(x_t, t) \) maps from state and period to a set of choices, i.e.,

\[
z^* \left( x_t, t \right) = \arg \max_{z_t} E_t \left\{ u \left( z_t, x_t, \varepsilon_t, t \right) + V \left( x_{t+1}, t+1 \right) \right\}.
\]

Notation: “arg max” means the argument (i.e. value of \( z_t \)) that maximizes the function.

The policy function is a decision rule: if you wake up on the \( t^{th} \) morning and discover that you have an endowment of \( x_t \), then you can refer to the rule \( z^* \left( x_t, t \right) \), carry out the action and go back to bed.

### III. The types of DP problems

Numerical DP problems can be broken into four basic types of problems, each requiring slightly different approaches.

#### A. Discrete state and discrete control (DD)

Discrete-Discrete problems (DD-DP problems) are problems for which computers are best suited. In such cases both the state variable and the control options take on a finite number of values. The inventory control problem presented in Lecture 7 is a DD problem. There are a finite number of states (i.e., inventory takes on only discrete values)
and the decision-maker's problem is to choose from a discrete set of possible production choices.

Because both the state space and the control space can be exactly modeled by the computer, there is no approximation error in the solution and it is possible to obtain an exact solution.

**B. Continuous state and discrete control (CD)**

In Continuous-Discrete problems (CD-DP problems) the state space is continuous, but there are only a finite number of choice possibilities. One example of this would be what are generally referred to as optimal-stopping problems. For example, the seminal paper by Rust (1987) considered the problem of the optimal time to replace the engine of a bus. The main state variable in Rust's model was the age of the current engine, a continuous variable. The choice was whether to replace the engine entirely or continue to give it maintenance. Hence, the state variable is continuous, the age of the engine, while the choice variable is discrete, replace or not.

CD problems are common in economics and are particularly well-suited for dynamic programming. Unlike in DD problems, in CD problems it is not possible to find the value of \( V \) at every point in the state space because there are an infinite number of such points. Further, we do not have a closed-form expression of \( V \) that would allow us to exactly calculate \( V \) at any point. We have to approximate. One way to solve CD problems is to calculate \( V(x) \) at a finite number of points and then estimate the value elsewhere in the state space based on the points where the solution has been found. Hence, the best we can do is to find an approximate solution to the problem. We'll discuss this problem in much more detail in Lecture 11.

**C. Continuous state and continuous control**

The third class of problems is where both the control and the state variables are continuous, CC-DP problems. In these problems both the state and control variable are continuous. The PV utility maximization problem discussed above is a good example of such a problem. In this case there are two sources of error – the approximation error in finding the correct choice variable from the continuous set of possibilities and the approximation error because \( V \) can only be estimated. Computer programs like GAMS or Matlab have embedded algorithms that are quite good at finding the solution of a static optimization problem. However, even in the best of worlds the solution will frequently only be approximate (though the error might be economically insignificant) and, if the problem is not concave, the program may find a solution that is entirely incorrect.

**D. Discrete state and continuous control (DC)**

The 4th class of problems are discrete state and continuous control (DC). This can arise, for example when a piece of machinery is either working or not and the question is how much to spend on maintenance each period. The solution algorithms for such problems are often an amalgam of DD and CD problems, so we will not spend much time considering this type of problem independently, though you should be able to solve such problems. There are problems in which there is a closed form expression for the optimal
choice in \( t \) as a function of the values of \( V_{t+1} \). In cases like that, the exact solution can be found without numerical approximation.

IV. A general algorithm for the solution of finite-horizon DD problems using a grid search method.

In DD problems, the state space is made up of a finite number of points, say \( X \). We'll assume that future benefits are discounted using the factor \( \beta \). If discounting is not required, then you'd simply have \( \beta=1 \).

There are three basic steps required

1) Initialization: set \( t=T+1 \) and define \( V(x_{T+1}, T+1) \) for every state \( x \in X \). If you don't have a salvage value, set \( V(x_{T+1}, T+1)=0 \) for all \( x_{T+1} \).

2) Recursion step: Starting with \( t=T \) find the value of \( z \) that solves the problem

\[
V(x, t) = \max_{z} E \left\{ u(z, x, t) + \beta V(x_{t+1}, t + 1) \right\},
\]

for every value of \( x \), then store \( V(x, t) \) and

\[
z^*(x, t) = \arg \max_{z} E \left\{ u(z, x, t) + \beta V(x_{t+1}, t + 1) \right\}
\]

3) Termination step: if \( t=0 \), stop. Otherwise set \( t=t-1 \) and return to step 2.

The pseudocode presented below gives a general representation of the solution algorithm, which could then be programmed up in almost any language that you choose. An implementation of this using Visual Basic is provided toward the end of these notes.
Pseudocode for a finite-horizon DD DP program

\[
\max_{z_t} E_0 \sum_{t=0}^T u(z_t, x_t, \epsilon, t) + V(X_{t+1}, T+1)
\]

Calculate \(V(x_{T+1}, T+1)\) for every state \(x \in X\) based on prior information.

For every period \(t = T, T-1, T-2, \ldots, 0\)

- for every point in your state space, \(x_t \in X\)
  - set \(V(x_t, t) = -\infty\) (in practice, a big negative number such as -999999)
  - for each possible decision choice, \(z_t\)
    - calculate \(E_t u(\cdot) = \sum_{\epsilon_t} p(\epsilon_t) u(z_t, x_t, \epsilon_t, t)\), where \(p(\epsilon)\) is the probability that a particular value of \(\epsilon\) will be realized
    - calculate \(E_t V(\cdot) = \sum_{\epsilon_t} p(\epsilon_t) V(x_{t+1}(z_t, x_t, \epsilon_t, t), t+1)\) where \(x_{t+1}(z_t, x_t, \epsilon_t, t)\) is the value for the state variable in \(t+1\) given \(z_t\) and a particular realization of \(\epsilon_t\)
    - calculate \(V' = E\{u(\cdot) + \beta V(\cdot)\}\) \((V'\) is a candidate value)
      - if \(V' > V(x_t, t)\) then
        - \(V(x_t, t) = V'\)
        - \(z_t^*(x_t, t) = z_t\)
      - EndIf
    - continue \{End of Control loop\}
  - continue \{End of Stage loop\}
- continue \{End of State loop\}

As you write your programs, each of these steps and the three main loops should be clearly identified. The control loop is where Bellman’s equation is solved at each state-stage combination.

V. Infinite-horizon dynamic programming problems

The finite-horizon problem that we consider above starts with some salvage value, \(V(x_{T+1}, T+1)\), and then works backwards. Hence, you might wonder how one might solve an infinite horizon problem, where there is no end point. It turns out, however, that because of discounting you still use the backward recursion solution approach. In effect, what we will do is start far out in the future and then work our way back to the present until, because of discounting, that starting value function far in the future, has no effect on what is done in the present.

Consider yourself in a situation in which you have a resource that you can use up to generate immediate utility or, with effort, you can increase the resource stock. If you had 100 units of the resource and only a few days to use it, then your choices today would be
very much affected by the time horizon. Your actions on day 3 would be very much
affected by the fact that the end of your problem is imminent. And if, on that day
somehow you were given an extra day to use the resource, your choices would probably
change a lot. Now imagine the same problem, but assume that you have 100 years to use
the resource. It is unlikely that your choices today would differ very much if someone
told you that you had 100 years + 1 day. If that’s true, if there is no change in your
choices today as the time horizon changes, you are behaving in the current period, \( t=0 \), as
if the time horizon doesn’t matter, i.e. as if you face an infinite time horizon. This
intuition is essentially what we implement in the successive approximation algorithm.

A. **Successive approximation of infinite-horizon problems**

We now want to consider how we would solve DP problems that don’t have a terminal
time, i.e. infinite-horizon problems. As indicated above, the basic intuition behind the
infinite-horizon Bellman’s equation is that, because of discounting, if \( T \) is much larger
than \( t \), then the function \( V(x,t) \) will look very much like \( V(x,t+1) \), and likewise for \( z^*(x,t) \)
and \( z^*(x,t+1) \). The solution algorithm converges if as \( T \to \infty \), \( V(x,t+1)=V(x,t) \). What this
means is that the value function will not change over time. Hence, we can drop our
reference to \( t \) and write Bellman’s equation:

\[
V(x) = \max_{z_t} \left\{ u(z_t, x_t, \epsilon_t) + \beta V(x_{t+1}) \right\}.
\]

Note that in this equation we have also dropped the reference to \( t \) in the benefit function
\( u(\cdot) \). This is a limitation – if you want to consider infinite-horizon DP problems you have
to specify your objective function in as autonomous. For many problems this is not a
limitation because they can be made autonomous by including additional state variables.
Equation 3 also includes \( \beta = 1/(1+r) \) in the general specification of the Bellman’s
equation. This is important; \( \beta<1 \) ensures that the objective function is finite, and will play
a central role in the numerical solution of the problem. The proof of convergence of the
successive-approximation algorithm requires that \( \beta \) be less than 1. Some undiscounted
problems converge, but typically they will not.

Now we have a dilemma. In the finite-horizon case, the function \( V(x_{r+1},T+1) \), being the
salvage value, was known. In the infinite-horizon case, however, we’re trying to solve for
an unknown function \( V(\cdot) \) on the LHS, but it also appears on the RHS. The starting point
is not obvious.

It turns out that there’s an easy solution to this problem. We can take any arbitrary guess
at the value function, say \( V^0(x) \), then using this function, solve for \( V^1(x) \) as follows

\[
V^1(x) = \max_{z_t} \left\{ u(z_t, x_t, \epsilon_t) + \beta V^0(x_{t+1}) \right\}.
\]

Then we repeat this process, finding \( V^2(x) \), \( V^3(x) \), ..., until we reach a point when
\( V^{k+1}(x) \) is “close” to \( V^k(x) \). Because we are only going to be interested in using the \( k^{th} \) and

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2 (at least for all \( t \) greater than some finite number)
the $k+1$\textsuperscript{th} approximation of the value function, we do not need to store all of our approximations of the value functions. Instead, at any time we only need to retain two value functions, which I will call $V^{\text{LHS}}(x)$ which is the new estimate and $V^{\text{RHS}}(x)$ which is the previous estimate.

\textbf{B. The successive approximation algorithm for the solution of infinite-horizon DD problems}

The successive approximation algorithm used to solve infinite-horizon problems is as follows:

As in the finite-horizon case, there are three basic steps required.

1) Initialization: For every state $x \in X$ take a first guess at $V(x)$, say $V^{0}(x)$. You can use $V^{0}(x) = 0$ for all $x$ or some other value if you have prior information about the solution. We will put $V^{0}(x)$ into the array $V^{\text{RHS}}(x)$.

2) Update step: Given our estimate $V^{\text{RHS}}$, obtain a new vector of values for $V^{\text{LHS}}(x)$ at every point $x \in X$ by solving the problems

$$V^{\text{LHS}}(x_t) = \max_{z_t} \mathbb{E}\left\{u(z_t, x_t, e_t) + \beta V^{\text{RHS}}(x_{t+1})\right\}.$$ 

Store $V^{\text{LHS}}$ and the candidate policy function,

$$\hat{\pi}(x_t) = \arg \max_{\hat{z}_t} \mathbb{E}\left\{u(z_t, x_t, e_t) + \beta V^{\text{RHS}}(x_{t+1})\right\}.$$ 

3) Convergence check: Check to see if

$$\max_{x} \left\| V^{\text{LHS}}(x) - V^{\text{RHS}}(x) \right\| < \tau,$$

where $\tau$ is a small convergence criterion.

\begin{itemize}
  \item If the convergence criterion is satisfied, then set $V(x) = V^{\text{LHS}}(x)$ and stop.
  \item Otherwise set $V^{\text{RHS}}(x) = V^{\text{LHS}}(x)$, and return to step 2 to find a new function, $V^{\text{LHS}}(x)$.
\end{itemize}
Here is the pseudocode for an infinite-horizon deterministic DD problem using a grid-search algorithm.

Set $V_{RHS}(x) = 0$ (or some other starting value) for every state $x \in X$.

For iteration $1, 2, \ldots, \text{max iter}$

- for every point in your state space, $x_t \in X$
  - set $V_{LHS}(x_t) = -\infty$ (in practice, a big negative number)
  - for each possible decision choice, $z_i$
    - $\tilde{u} = u(z_i, x_t)$
    - $\tilde{V} = V_{RHS}(x_{t+1}(z_i, x_t))$
    - [We now have “candidate” values $u(\cdot)$ and $V(x_{t+1})$ for a given choice]
    - if $\tilde{u} + \beta\tilde{V} > V_{LHS}(x_t)$ then
      - $V_{LHS}(x_t) = \tilde{u} + \beta\tilde{V}$
      - $z^*(x_t) = z_i$
    - endif
  - continue [end of control loop]
- continue [end of state loop]

set $\text{diff} = \max_x \left| V_{LHS}(x) - V_{RHS}(x) \right|$  

set $V_{RHS}(x) = V_{LHS}(x)$

if $\text{diff} \leq \text{convergence criterion}$, then exit stage loop

endif

continue [end of stage loop, start next iteration with a new $V_{RHS}$]

set $V(x) = V_{LHS}(x)$ for all $x$.
And here is the pseudocode for an infinite-horizon stochastic DD problem.

Set $V^{\text{RHS}}(x) = 0$ (or some other starting value) for every state $x \in X$.

For iteration 1, 2, ..., max iter

for every point in your state space, $x_t \in X$

set $V^{\text{LHS}}(x_t) = -\infty$ (in practice, a big negative number)

for each possible decision choice, $z_t$

set $E_t \mu(\cdot) = E_t V(\cdot) = 0$

for each state of nature, $\varepsilon_t$ with probability $p(\varepsilon_t)$

evaluate $u(z_t, x_t, \varepsilon_t)$

evaluate $V(x_{t+1}(z_t, x_t, \varepsilon_t))$ where $x_{t+1}(z_t, x_t, \varepsilon_t)$ is the state value that occurs in $t+1$ given a particular action $z_t$ and a particular realization of $\varepsilon_t$

Add $p(\varepsilon_t) u(z_t, x_t, \varepsilon_t)$ to $E_t \mu(\cdot)$

Add $p(\varepsilon_t) V(x_{t+1}(z_t, x_t, \varepsilon_t))$ to $E_t V(\cdot)$

continue [end of uncertainty loop]

[We now have "candidate" values $E_t \mu(\cdot)$ and $E_t V(\cdot)$ and

$\tilde{V} = E_t \mu(\cdot) + \beta E_t V(\cdot)$ for a given choice]}

if $E_t \mu(\cdot) + \beta E_t V(\cdot) > V^{\text{LHS}}(x_t)$ then

$V^{\text{LHS}}(x_t) = E_t \{u(\cdot) + \beta V^{\text{RHS}}(\cdot)\}$

$z^*(x_t) = z_t$

endif

continue [end of control loop]

continue [end of state loop]

set $\text{diff} = \max_x \left| V^{\text{LHS}}(x) - V^{\text{RHS}}(x) \right|$

set $V^{\text{RHS}}(x) = V^{\text{LHS}}(x)$

if $\text{diff} <$ convergence criterion, then exit loop and set $V(x) = V^{\text{LHS}}(x)$, else

continue [end of stage loop]

The reason that this algorithm converges to the correct $V(x)$ is because the Bellman’s equation is a contraction mapping. What this means is that the distance between $V^{\text{LHS}}$ and $V^{\text{RHS}}$ decreases monotonically over iterations. Hence, if you have a maximum difference between the two value functions of 0.01 after 20 iterations, then even if you iterate 1000 more times, you will never see a maximum difference of more than 0.01.
Technical details and a proof of that this algorithm is a contraction mapping can be obtained from a variety of sources including the books by Bertsekas, Judd, and Miranda and Fackler.

The proof of the contraction mapping property gives rise to an interesting result. If \( V^k \) and \( V^{k-1} \) are two successive iterations of the value function, then we can place bounds on the true infinite-horizon value function using these two functions as follows. Define

\[
\frac{\beta}{1 - \beta} \min \left[ V^k - V^{k-1} \right] \quad \text{and} \quad \frac{\beta}{1 - \beta} \max \left[ V^k - V^{k-1} \right].
\]

Then if \( V \) is the true infinite-horizon value function, then

\[
V^k + b_k \leq V \leq V^k + \bar{b}_k \quad \text{(Rust, 1996 p. 653).}
\]

That is, there are bounds between which we know the true value function will actually lie. Furthermore, these bounds will get tighter as the number of iterations increases. Because that can be shown, it follows that the successive approximation algorithm will converge to the true value function.

Here are a couple of things worth highlighting about the infinite-horizon problem:

- The Bellman’s equation of the infinite-horizon problem is what is known as a **functional fixed point.** In the case of DD problems the function \( V(\cdot) \) is simply a set of values for each value \( x_t \in X \). At the fixed point, when we evaluate the Bellman’s

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3 I have found instances when this formula seems to break down, but I have not explored it in detail and I cannot rule out programming errors on my end. Regardless, as \( k \to \infty \), the formula certainly holds.

4 As will be discussed in Lecture 12, these bounds to accelerate the solution algorithm.
equation to find $V^{LHS}(x_i)$, we will find the same vector of values as those in $V^{RHS}(x_i)$. Note that this does not mean that $V^{LHS}(x_i) = V^{RHS}(x_{i+1}(x_i, z_i))$.

- The successive approximation algorithm is monotonic in the sense that if $V^{k+1}(x) > V^k(x)$ for all $x \in X$, then in every subsequent iteration for $n \geq k + 1$, $V^{n+1}(x) > V^n(x)$. This property is critical to the convergence of the algorithm. As a practical manner, you can sometimes use this property in your programming – if it is violated you have an error in your program. (Note that this inequality holds point by point!)

- In the successive approximation algorithm there is no need to store the value function at each stage since in the end all you are looking for is the true value function $V(x)$. For debugging purposes, however, it is sometimes helpful to save them.
VI. DP with VB

On the next several pages we present two Visual Basic programs that solve the inventory control problem presented in the previous lecture. In the first case, note the close correlation between the pseudocode and the actual program. The second program is more sophisticated and general. Either program can be modified to solve virtually any finite-horizon DD-DP problem. Spreadsheets with these programs are available on the programs page of the class web site.

**BASIC VB Code for solving the Inventory control problem**

```vbnet
Option Explicit
Option Base 1

Sub VerySimpleDP()
    ' Dimension the minimal set of variables needed
    Dim it As Integer, ix As Integer, ixnext As Integer, iz As Integer
    Dim xt As Double, xnext As Double, zt As Double
    Dim V(0 To 3, 1 To 4), zStar(0 To 3, 1 To 4)
    Dim u, VLHS, VRHS
    ' Parameters of the model
    Dim p, d, alpha, gamma, N
    N = 3
    p = 10
    d = 2
    alpha = 1
    gamma = 3
    ' ---------------------------------------------------------
    ' Terminal Values. Sell product and pay fine if required
    it = 4
    For ix = 0 To 3
        xt = ix*1.0
        V(ix, it) = (p * xt - d * (N – xt) )
        Next ix
    ' ---------------------------------------------------------
    ' Recursion step starting in T-1
    ' Stage Loop, stepping backward
    For it = 3 To 1 Step -1
        ' -------------------------------------------------------
        ' Start State Loop
        For ix = 0 To 3
            xt = ix*1.0
            ' -------------------------------------------------------
            ' Start by setting V(ix,it) = -99999
            V(ix, it) = -99999
            ' ---------------------------------------------------
            ' Start Control Loop
            For iz = 0 To 3
                zt = iz*1
                ' -----------------------------------------------
                ' Evaluate the candidate policy. VLHS = u + VRHS
                u = -(alpha * zt ^ 2 + gamma * xt)
                xnext = xt + zt
                ' State Equation
                If xnext > N Then xnext = N ' boundary issues
                iznext = xnext
                ' Get index for xnext
                VRHS = V(iznext, it + 1)
                VLHS = u + VRHS
                ' Check to see if this z is an improvement.
                If VLHS > V(ix, it) Then
                    V(ix, it) = VLHS
                    zStar(ix, it) = zt
                End If
                ' ---------------------------------------------
                ' End of control loop
            Next iz
            ' -------------------------------------------------
            ' End of state loop
        Next ix
    ' ---------------------------------------------------
    ' Store output in Sheet 1.
    ' use 5 - ix & 15-ix so that results are from highest x to lowest as in notes
    Sheets("Sheet1").Select
    For ix = 0 To 3
        For it = 1 To 4
            Cells(5 - ix, it) = V(ix, it)
            Cells(15 - ix, it) = zStar(ix, it)
        Next it
    Next ix
End Sub
```
Sub Main()
    Call initializeValues
End Sub

' CreateGrids of for x and z
' The CreateGrid Function is available in the MatrixSubs that
' can be downloaded from the web site
Dim xGrid = CreateGrid(0, nx, xMin, xMax)
Dim zGrid = CreateGrid(0, nz, zMin, zMax)

' Set Terminal Value [or the initial estimate of V() in infinite-horizon problems]
Call TerminalValue
Store terminal values in ValFn and on the spreadsheet
For ix = 0 To nx
    ValFn(ix, nT) = VRHS(ix, 1)
    Range("ValueFunction").Offset(ix + 1, nT).Value = ValFn(ix, nT)
Next ix

' stage loop (note we start at it=nT and move backwards until it=1
For it = (nT - 1) To 1 Step -1
    ' state loop
    For ix = 0 To nx
        xt = xGrid(ix)
        VLHS(ix, 1) = -99999#
        ' -----------------------------------
        ' Solve the maximization problem at current state & stage
        ...          ' -----------------------------------
        ' Compare stored V with V at current z and update if necessary
        If V > VLHS(ix, 1) Then
            VLHS(ix, 1) = V
            zStar(ix, it) = zt
        End If
    Next ix
    ' -----------------------------------
    '   End of control loop
Next it

' This program solves the inventory control problem that is solved
' By hand in AGEC 637
' Author: Rich Woodward
' http://agecon2.tamu.edu/people/faculty/woodward-richard/642/Programs/
' When basing problem-set programs on this code, you need to rewrite
' and add comments to convince me that you fully understand
' the program.

' VARIABLE DECLARATIONS & Global options
Option Explicit ' All variables used must be explicitly dimensioned
Option Base 1   ' Unless specified otherwise, arrays will be indexed
                ' starting at 1. You override this by writing,
                ' for example,  “ Dim V(0 To nx) ”

' Notice that all variables are dimensioned outside the subs so that their
' values are accessible from all the subs.
' Arrays are dimensioned with the () after them so that they can
' be redimensioned later after dimension parameters have been read in
Dim ValFn() As Double ' Final Value Function
Dim VLHS() As Double  ' Value Function on LHS of Bellman's eq
Dim VRHS() As Double  ' Value Function on RHS of Bellman's eq
Dim zStar() As Integer '  Optimal policy function of xt and it

' Parameters of the program
Dim nT As Integer, nx As Integer, nz As Integer

' Parameters of the model
Dim Beta As Double, p As Double, d As Double, n As Double
Dim alpha As Double, gamma As Double

' Intermediate variables
Dim utility As Double, V As Double, Vnext As Double

' Counters. Some variables require both a real # and integer representation.
Dim it As Integer, iz As Integer, ix As Integer, ixnext As Integer

' x and z are potentially real numbers that will be taken from the
' grids xGrid and zGrid
Dim zt, xt, xnext
Dim zMin, zMax, xMin, xMax
Dim xGrid, zGrid
' Store optimal value in the full array of the value function
ValFn(ix, it) = VLHS(ix, 1)
' Store optimal values in spreadsheet
Range("ValueFunction").Offset(ix + 1, it).Value = ValFn(ix, it)
Range("Zstar").Offset(ix + 1, it).Value = zStar(ix, it)
Next ix

' Move VLHS over to the RHS
' In infinite-horizon problems at this point
' check convergence, and carry out the policy-iteration or
' modified-policy-iteration step
For ix = 0 To nx
VRHS(ix, 1) = VLHS(ix, 1)
Next ix

' End of Stage loop
Next it

' The optimization problem is solved!
' Now simulate the optimal policy path
Call Simulation

' End of main program
End Sub

Sub UtilityFunction()
' The benefit function subroutine as a function of z and x
utility = -alpha * (zt / 100) ^ 2 - gamma * (xt / 100)
End Sub

Sub StateEquation()
' The state equation subroutine
xnext = xt + zt
' You must decide what happens if you go outside your state grid.
' This can often be a critical modelling decision
If xnext > xGrid(nx) Then xnext = xGrid(nx)

Sub TerminalValue()
For ix = 0 To nx
xt = xGrid(ix)
VRHS(ix, 1) = p * xt / 100 - d * (n - xt) / 100
If xt > n Then VRHS(ix, 1) = p * n / 100
Next ix
End Sub

Sub VnextCalc()
' Subroutine to calculate V(x+1, t+1)
Vnext = VRHS(ixnext, 1)
End Sub

Sub initializeValues()
' This subroutine sets up the space for writing output
' and reads in the data
Clean up the area for writing output
Call ClearAreas

Read in data from spreadsheet
nT = Range("nT")
nx = Range("nx")
nz = Range("nz")
p = Range("p")
d = Range("d")
n = Range("n")
alpha = Range("alpha")
gamma = Range("gamma")
Beta = Range("beta")

Set up maxes & mins for the state and control variables
zMin = 0
zMax = nz * 100
xMin = 0
xMax = nx * 100
Redimension arrays based on read in data
When you redimension an array you set its value to zero. Be careful!
Note that in the inventory problem we want the arrays to start at zero
ReDim ValFn(0 To nx, nT)
ReDim VLHS(0 To nx, 1)
ReDim VRHS(0 To nx, 1)
ReDim zStar(0 To nx, nT)

Set up areas for writing output
Range("output"), Offset(2, 0).Name = "ValueFunction"
Range("ValueFunction"), Offset(nx + 4, 0).Name = "Zstar"
Range("Zstar"), Offset(nx + 4, 0).Name = "Simt"
Range("Simt"), Offset(1, 0).Name = "SimXt"
Range("SimXt"), Offset(1, 0).Name = "SimZt"
Range("SimZt"), Offset(1, 0).Name = "SimXtPlus1"
Range("SimXtPlus1"), Offset(1, 0).Name = "SimUtility"

Add titles to the output arrays
Range("ValueFunction"), Value = "V"
Range("Zstar"), Value = "Zstar"
For it = 1 To nT
  Range("ValueFunction"), Offset(0, it).Value = "t = " & it
  Range("Zstar"), Offset(0, it).Value = "t = " & it
Next it
For ix = 0 To nx
  Range("ValueFunction"), Offset(ix + 1, 0).Value = "x = " & ix
  Range("Zstar"), Offset(ix + 1, 0).Value = "x = " & ix
Next ix
Range("Simt"), Offset(-1, 0).Value = "Simulation"
Range("Simt"), Value = "t"
Range("SimXt"), Value = "x"
Range("SimZt"), Value = "z"
Range("SimXtPlus1"), Value = "xt+1"
Range("SimUtility"), Value = "Utility"

Sub Simulation()
Now simulate the path over nT periods
xt = 0
For it = 1 To nT
  Find the index associated with the real # xt
  The InvertGrid Function is available in the MatrixSubs that can be downloaded from the web site
  ix = InvertGrid(xt, xGrid)
  Choose the optimal choice z* from the zstar array
  zt = zStar(ix, it)
  figure out what x(t+1) will be given z*
  Call StateEquation  if finds xnext as fn of xt zt
  Call UtilityFunction
  write the output to the spreadsheet
  Range("Simt"), Offset(0, it).Value = it
  Range("SimXt"), Offset(0, it).Value = xt
  Range("SimZt"), Offset(0, it).Value = zt
  Range("SimXtPlus1"), Offset(0, it).Value = xnext
  Range("SimUtility"), Offset(0, it).Value = utility
  Update to the next period
  xt = xnext
  end of simulation loop
Next it
End Sub

Sub ClearAreas()
This program may be used directly without personalized comments
Clear space for printing value function
With Range("OUTPUT")
  Range(Offset(1, 0), Offset(100, 50)).ClearContents
End With
End Sub