I. Positive analysis using a DP foundation: Econometric applications of DP

One of the most active areas of research involving numerical dynamic optimization, is the use of DP models in positive analysis. This work seeks to understand the nature of a particular problem being solved by a decision maker. By specifying explicitly the optimization problem being solved, the analyst is able to estimate the parameters of a structural model, as opposed to the easier and more common reduced form approach. This approach was first developed by John Rust (1987) and I draw directly on his original paper to explain how this is done. He provides a more complete description of his approach in Rust (1994a and 1994b).

My focus here will be quite narrow and serves only as an introduction to this literature. A more general and up-to-date review of approaches for the estimation of the parameters of dynamic optimization problems is provided by Keane et al. (2011).

Recall that Rust’s paper was “Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher.” The state variable $x_t$ denotes the accumulated mileage (since last replacement) of the GMC bus engines of the bus fleet in the Madison, Wisconsin. Harold Zurcher must make the choice as to whether to carry out routine maintenance, $i=0$, or replace the engine, $i=1$. Each period the operating costs, $c(x_t, \theta)$ where $\theta$ is a vector of parameters to be estimated. He assumes that mileage traveled in a period, say $\Delta x$, is distributed exponentially so that the cdf of $\Delta x$ is $F(x, \theta) = 1 - \exp(-\theta \Delta x)$. If you replace the engine, $i=1$, then $x_{t+1} = x_t + \Delta x$, and if the engine is not replaced, $i=0$, then $x_{t+1} = x_t$.

An old engine has scrap value $\overline{P}$ and a new one costs $\overline{P}$. Hence, the additional cost to replace an engine is equal to $(\overline{P} - P)$.

The problem he is solving, therefore, is what Rust calls a regenerative optimal stopping problem:

$V_\theta(x_t) = \max_{i=0,1} \left[ u(x_t, i, \theta) + \beta EV_\theta(x_{t+1}, i) \right]$ (3.5)

where

$EV_\theta(x_t, i) = \int_0^\infty V_\theta(y) p(y | x_t, i, \theta) \, dy$ (3.6)

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1 Adda and Cooper’s textbook focuses particularly on this application of DP and the interested reader is referred to them for an introduction.

2 I follow Rust’s notation here. Following the notation more common in this series of lecture notes, $EV_\theta(x_t, i)$ would be written $EV_\theta(x_{t+1}, i)$, $x_{t+1} = f(x_t, i)$.
where \( y \) is the index of the integral, i.e. what is used in place of \( x \) inside the integral, and \( p(y) \) being a density function, i.e. \( \partial F(y)/\partial y \). The econometric task is to estimate the parameter vector \( \theta \).

As Rust notes, the problem above has much in common with standard models of discrete choice in which the econometric problem is to identify the parameters \( \theta \) that maximize the likelihood of observing the set of data. That is, if we let

\[
\bar{V}_\theta(x_i) = u(x_i, i_i) + \beta EV_\theta(x_i, i_i)
\]

i.e., the value function with the choice \( i_i \), then, for a given value for the parameter vector \( \theta \), we can write the probability of observing \( i_i = 1 \), as the probability that

\[
\bar{V}_\theta(x_i, 1) > \bar{V}_\theta(x_i, 0).
\]

This is essentially the same econometric problem as considered by McFadden (1973). In standard structural estimation of discrete choice models, however, the objective function is a static optimization problem that can be evaluated quickly. In a dynamic problem the function \( \bar{V}_\theta(x_i, 1) \) is the solution to a dynamic optimization problem.

**A. The approach he doesn’t like**

Using the Bellman’s equation, ... there is an optimal stationary, Markovian replacement policy \( \Pi = (f, f, ... ) \) where \( f \) is given by

\[
i_i = f(x_i, \theta) = \begin{cases} 1 & \text{if } x_i > \gamma(\theta_1, \theta_2) \\ 0 & \text{if } x_i \leq \gamma(\theta_1, \theta_2) \end{cases}
\]

where \( \gamma(\theta_1, \theta_2) \) is the solution to

\[
\beta(P - P) = \frac{1}{r} \int_0^{\gamma(\theta_1, \theta_2)} \left[ 1 - \beta \exp\{-\theta_2(1 - \beta) y\} \right] \frac{\partial c(y, \theta)}{\partial y} dy
\]

where \( \gamma \) represents a threshold value of mileage. The left hand side is the cost of replacing the engine in the next period, a cost that is assumed to fall in the next period, and the right-hand side is the capitalized value of the stream of expected savings that result from having a new engine.

He goes on to state:

“The likelihood function \( l(i_1, ..., i_T, x_1, ..., x_T, \theta) \) specifies the conditional probability density of observing the sequence of states and replacement decisions for a single bus in periods 1 to \( T \). Under the assumption that monthly mileage and replacement decisions are independently distributed across buses, the likelihood function \( L(\theta) \) for the full sample of data is simply the product of the individual bus likelihoods \( l \). “(p. 1007)

This estimation approach, however, depends critically on functional form assumptions that are unlikely to hold. As Rust writes, “The solution of the likelihood function depends critically on specific choice of functional form: namely, that monthly mileage
$(x_{t+1} - x_t)$ has an i.i.d. exponential distribution. Unfortunately, my sample of data flatly refutes this assumption.”

The more general problem faced is provided in paragraphs on pages 1008-1009:

A basic result in Markovian decision theory (cf. Blackwell (1968)) shows that under quite general conditions the solution to the class of infinite horizon Markovian decision problems takes the general form

\begin{equation}
  \pi_t = f(x_t, \theta)
\end{equation}

where $f$ is some deterministic function relating the agent’s state variables $x_t$ to his optimal action $i_t$. Suppose we assume that there are no unobserved state variables, i.e. that the econometrician observes all of $x_t$. The theory then implies that the data obey the deterministic relation (3.9) for some unknown parameter value $\theta^*$. However in general, real data will never exactly obey (3.9) for any value of the parameter $\theta$: the data contradict the underlying optimization model. The typical solution to this problem is to “add an error term” $\epsilon_t$ in order to reconcile the difference between $f(x_t, \theta)$ and the observed choice $i_t$.

\begin{equation}
  i_t = f(x_t, \theta) + \epsilon_t.
\end{equation}

By making a convenient distributional assumption for $\epsilon_t$, one might use the model (3.10) to estimate $\theta$. The difficulty with this procedure is that it is internally inconsistent: the structural model was formulated on the hypothesis that the agent’s behavior is described by the solution of a dynamic optimization problem, yet the statistical implementation of that model implies that the agent randomly departs from this optimal solution. If error terms $\epsilon_t$ are to be introduced to a structural model in an internally consistent fashion, they must be explicitly incorporated into the solution of the dynamic optimization problem. When this is done, a correct interpretation of the “error term” $\epsilon_t$ is that it is an unobservable, a state variable which is observed by the agent but not by the statistician.

B. The approach he does like

Rust (1987) proposes an alternative structural model. He notes that Zurcher’s Bellman’s equation is

\begin{equation}
  V_\theta(x_t, \epsilon_t) = \max_i \left[ u(x_t, i, \theta) + \epsilon_t(i) + \beta EV_\theta(x_{t+1}, \epsilon_t, i) \right],
\end{equation}

which is a function of the distance that the engine has traveled, $x_t$, and an unobserved state variable, $\epsilon_t$, which is known by Zurcher but not by the analyst. Writing out the expected value on the RHS of the Bellman’s equation, we have

\begin{equation}
  EV_\theta(x_t, \epsilon_t, i) = \int \int V_\theta(y, \eta) p(dy, d\eta \mid x_t, \epsilon_t, i, \theta_2, \theta_3).
\end{equation}

The function $p(\cdot)$ is a continuous Markov transition density function, which allows us to estimate the expected value of $x_{t+1}$ conditional on the choice $i$, the current state, $x_t$, and the parameters $\theta_2$ and $\theta_3$.

Rewriting (4.4) in way more familiar to us, we obtain
(4.4') \[ V_\theta(x_i, \epsilon) = \max_{i} \left[ u(x_i, i, \theta) + \epsilon(i) + \beta EV_\theta(x_{i+1}, \epsilon_{i+1}) \right]. \]

Hence, Rust’s preferred approach involves having the unobserved state variable, \( \epsilon \), enter linearly into the benefit function. As we can see in (4.5), however, there is the possibility that \( \epsilon \) will be correlated across time since the density function for \( x_{t+1} \) and \( \epsilon_{t+1} \), is conditional on \( x_t \) and \( \epsilon_t \). That would be a problem because the distribution at \( t+1 \), is a function of the variable in \( t \), which is itself unknown. Hence, the distribution grows geometrically as we look from \( t \) to \( t+1 \), to \( t+2 \) and so on, making the problem computationally intractable.

To achieve an estimable model, therefore, Rust makes a critical assumption of Conditional Independence (CI) (p. 1011). This assumption “implies that any statistical dependence between \( \epsilon_t \) and \( \epsilon_{t+1} \) is transmitted entirely through the vector \( x_{t+1} \). Second, the probability density of \( x_{t+1} \) depends only on \( x_t \) and not \( \epsilon_t \).”

As an important practical matter, the CI assumption means that the transition density, can be written in a multiplicative form

\[ p(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t, i, \theta_2, \theta_3) = q(\epsilon_{t+1} | x_{t+1}, \theta_2) p(x_{t+1} | x_t, i, \theta_3), \]

where \( \theta_2 \) and \( \theta_3 \) are parameters governing the dynamics of the state variables \( \epsilon \) and \( x \) respectively. In other words, \( q(\cdot) \) and \( p(\cdot) \) are independent Markov transition probabilities. Note that neither \( q \) nor \( p \) are conditional on \( \epsilon_t \).

Making the CI assumption leads to his Theorem 1, which implies, (p. 1012) “that the conditional choice probabilities \( P(i|x, \theta) \) [i.e., the conditional probability of choosing action \( i \) given the state variable \( x \) ] can be computed using the same formulas used in the static case with the addition of the term \( \beta EV_\theta(x, i) \) to the usual static utility term \( u(x_i, i, \theta) \). Notice that McFadden’s (1973), (1981) static model of discrete choice appears as a special case of Theorem 1 when \( p(\cdot | x, i, \theta) \) is independent of \( i \).”

Following on McFadden, if we make the assumption that \( q(\epsilon|x, \theta_2) \) is multivariate extreme value, then the probability of observing choice \( i \), \( P(i|x, \theta) \), can be written as a multinomial logit:

\[ P(i | x, \theta) = \frac{\exp \left[ u(x_i, i, \theta) + \beta EV_\theta(x_{i+1}, (x, i)) \right]}{\sum_{j \in C(x)} \exp \left[ u(x_j, j, \theta) + \beta EV_\theta(x_{i+1}, (x, j)) \right]}. \]

The log likelihood function, therefore, can be written

\[ LL = \sum_{k=1}^{n} \ln P(i_k | x_k, \theta), \]

where \( i_k \) and \( x_k \) are the \( k^{th} \) observation observed in the data set.

Hence, Rust suggests “the following nested fixed point algorithm: an ‘inner’ fixed point algorithm computes the unknown function \( EV_\theta \) for each value of \( \theta \) and an “outer” hill
climbing algorithm searches for the value of $\theta$ which maximizes the likelihood function.”

That is:

1) Choose a vector of parameters of $\hat{\theta} = \theta^i$
2) Solve the DP to find $V_{\theta}$ with $\theta = \hat{\theta}$
3) Calculate the likelihood function for your data set (and slopes of the likelihood function w.r.t the elements of $\theta$)
4) Update $\hat{\theta} = \theta^i$ and return to step 1.

Since we will be iterating repeatedly on this, efficient solutions to the DP (step 2) is critical. Collocation methods (see lecture 11) are frequently used for this step.

Rust is not overly optimistic about the ability to apply this approach to a wide range of problems. The computational burden is great. Each time we calculate the Bellman’s equation it is necessary to solve an infinite horizon DP problem, and then you need an efficient algorithm for searching over the parameter space. He cites his own work which showed that the contraction mapping $T_{\theta}$ (i.e., the successive approximation algorithm) is Fréchet differentiable.\footnote{According Weisstein, a function is Fréchet differentiable at $a$ if $\lim_{x \to a} (f(x) - f(a))/(x-a)$ exists. That is Fréchet differentiability is the standard concept of continuous differentiability.}

This gives him two numerical advantages. He uses a method to obtain $EV_\theta$ and, as a by-product of this method, gets analytic solutions for the $\theta$ derivatives for $EV_{\theta}$, which can then be used by the hill-climbing algorithm to maximize the likelihood function. (p. 1013)

C. Applications

There have been many applications of Rust’s nested fixed-point approach and with the increasing speed of computers and the use of efficient solution methods, there is great scope for expanding the applications of the nested fixed point algorithm. A few papers by applied economists that have used these methods are Miranda and Schnitkey (1995), Provencher (1995), and Baerenklau and Provencher (2005).

An interesting extension of this approach was provided by Hendel and Nevo (2005) who consider the consumer problem of a storable good – laundry detergent.\footnote{Hendel and Nevo (2005) is a testament to the lengths that researchers have gone to understand the deep mysteries behind the demand for laundry detergent, which was one of the most important social issues of the last century.} In their application the problem is further complicated by the fact that they never observe the state variable, the stock of laundry detergent in the household, and they do not observe withdrawals from the stock either, only additions when the individual makes a purchase. The bottom line is that they do find substantial differences between the static and dynamic models, with static models overestimating demand elasticities by 30%.
D. Other methods and critiques

Although these notes focus on Rust’s nested fixed-point algorithm, his is not the only method that has been developed for estimating dynamic structural models. Other approaches include Keane and Wolpin (1994), Pakes et al., (2007) and Bajari et al., (2007, 2009). Rust (1994b) describes an alternative “forward” approach that “abandons the pretense of starting with an ‘a priori’ specification for \( u \) and instead conducts a specification search over the value functions \( v \) directly, using the estimated \( v \) functions to ‘back out’ estimates of the single-period utility functions \( u \) from the fixed point condition” (p. 149). This approach is computationally much less burdensome and he notes that in his 1988 paper, he finds that both methods generate “basically similar estimates of \( u \) and \( v \).”

The structural approach advocated by Rust, however, is not without its detractors. Heckman and Navarro (2007) describe Rust’s approach as follows:

[Rust (1994a)] shows that without additional restrictions, a class of infinite horizon dynamic discrete choice models for stationary environments is nonparametrically nonidentified. His paper has fostered the widespread belief that dynamic discrete choice models are identified only by using arbitrary functional form and exclusion restrictions. The entire dynamic discrete choice project thus appears to be without empirical content, and the evidence from it at the whim of investigator choices about functional forms of estimating equations and application of ad hoc exclusion restrictions.

Not surprisingly, they go on to offer an alternative.

II. References


