14. Optimal control with constraints and MRAP problems

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We now return to an optimal control approach to dynamic optimization. This means that our problem will be characterized by continuous time and will be deterministic.

It is usually the case that we are not *Free to Choose*.¹ The choice set faced by decision makers is almost always constrained in some way and the nature of the constraint frequently changes over time. For example, a binding budget constraint or production function might determine the options that are available to the decision maker at any point in time. When this is true we will need to reformulate the simple Hamiltonian problem to take account of the constraints. Fortunately, in many cases economic intuition will tell us that the constraint will not bind (except, for example, at \( t=T \)), in which case our life is much simplified. We consider here cases where we're not so lucky, where the constraints cannot be ruled out ex ante.

We will assume throughout that a feasible solution exists to the problem. Obviously, this is something that needs to be confirmed before proceeding to waste a lot of time trying to solve an infeasible problem.

In this lecture we cover constrained optimal control problems rather quickly looking at the important conceptual issues. For technical details, I refer you to Kamien & Schwartz, which has chapters on constrained optimal control problems. We then go on to consider a class of problems where the constraints play a particularly central role in the solution.

I. Optimal control with equality constraints

A. Theory

Consider a simple dynamic optimization problem

\[
\max_z \int_0^T e^{-\alpha t} u(z, x, t) \, dt \quad \text{s.t.} \quad \begin{align*}
\dot{x} &= g(z, x, t) \\
h(z, x, t) &\geq c \\
x(0) &= x_0
\end{align*}
\]

In this case we cannot use the Hamiltonian alone, because this would not take account of the constraint, \( h(z, x, t) \geq c \). Rather, we need to maximize the Hamiltonian subject to a constraint \( \Rightarrow \) so we use a Lagrangian² in which \( H \) is the objective function, i.e.,

\[
L = H + \phi(h(z, x, t) - c)
\]

\[
= u(z, x, t) + \mu g(z, x, t) + \phi(c - h(z, x, t)).
\]

Equivalently, you can think about embedding a Lagrangian, within a Hamiltonian, i.e.

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¹ This is an obtuse reference to the first popular book on economics I ever read, *Free to Choose* by Milton and Rose Friedman.

² This Lagrangian is given a variety of names in the literature. Some call it an augmented Hamiltonian, some a Lagrangian, some just a Hamiltonian. As long as you know what you’re talking about, you can pretty much call it any of these terms.
\[ H_c = u(z, x, t) + \phi(c - h(z, x, t)) + \mu g(z, x, t). \] We’ll use the first notation here.

Assuming that everything is continuously differentiable and that concavity assumptions hold, the FOC’s of this problem, then, are:

1. \( \frac{\partial L}{\partial z} = 0 \)

2. \( \frac{\partial L}{\partial x} = r\mu - \dot{\mu} \)

and, of course, the constraints must be satisfied:

\[ \frac{\partial L}{\partial \mu} = \dot{x} \]

\[ \frac{\partial L}{\partial \phi} = c - h(z, x, t) = 0. \]

Let’s look at these in more detail. The FOC w.r.t. \( z \) is

1’. \( \frac{\partial L}{\partial z} = \frac{\partial u}{\partial z} + \mu \frac{\partial g}{\partial z} - \phi \frac{\partial h}{\partial z} = 0, \)

which can be rewritten

1”. \( \frac{\partial u}{\partial z} - \phi \frac{\partial h}{\partial z} = -\mu \frac{\partial g}{\partial z}. \)

As Dorfman showed us, the FOC w.r.t. the control variable tells us that at the optimum we balance off the marginal current benefit and marginal future costs. In this case the RHS is the cost to future benefits of a marginal increase in \( z \). The LHS, therefore, must indicate the benefit to current utility from marginal increments to \( z \). If \( \frac{\partial u}{\partial z} > \text{RHS} \), then this implies that there is a cost to the constraint and \( \phi \frac{\partial h}{\partial z} \) is the cost to current utility of the intratemporal constraint, \( h \). If \( h(\cdot) \) were marginally relaxed, then \( z \) could be changed to push it closer to balancing with the contribution of a marginal unit of \( z \) in the future.
In principle, the problem can then be solved based on these equations. It is important to note that $\phi$ will be a function of time and will typically change over time. What is the economic significance of $\phi$?

**B. Optimal control with multiple equality constraints**

The extension to the case of multiple equality constraints, is easy; with $n$ constraints the Lagrangian will take the form

$$L = u(z, x, t) + \lambda g(z, x, t) + \sum_{i=1}^{n} \phi_i \left( h_i(z, x, t) - c_i \right).$$

Obviously, there may not be a feasible solution unless some of the constraints do not bind or are redundant, especially if $n$ is greater than the cardinality of $z$.

**C. Example: The political business cycle model (Chiang’s (Elements of Dynamic Optimization) presentation of Nordhaus 1975)**

This model looks at macroeconomic policy. Two policy variables are available, $U$, the rate of unemployment, and $p$, the rate of inflation. It is assumed that there is a trade-off between these two so that support for the current administration can be defined by the equation

$$v = v(U, p)$$

so that the relationship between the two policies can be described by the iso-vote curves in the figure below.

![More votes diagram](image)

Following standard Phillips-curve logic, there is an assumed trade-off between these two objectives so that the inflation rate goes down if the unemployment rate goes up,

$$p = \gamma(U) + \alpha \pi,$$

where $\pi$ is the expected rate of inflation. Expectations evolve according to the differential equation

$$\dot{\pi} = b(p - \pi).$$
We assume that the votes obtained at time \( T \) are a weighted sum of the support that is obtained from 0 to \( T \), with support nearer to the voting date being more important. Votes obtained at \( T \) are equal to \( \int_0^T v(U, p) e^\pi dt \).

The optimization problem then is
\[
\max_{U, p} \int_0^T v(U, p) e^\pi dt \quad \text{s.t.} \quad \dot{\pi} = b(p - \pi) \\
p = \gamma(U) + \alpha \pi \\
\pi(0) = \pi_0, \quad \text{and} \quad \pi(T) \text{ free.}
\]

Now clearly the equality constraint could be used to substitute out for \( p \) and convert the problem to a single-variable control problem, but let’s consider the alternative, explicitly including the constraint.

The Lagrangian (or augmented Hamiltonian) for this optimal control problem would be
\[
L = v(U, p) e^\pi + \lambda(b(p - \pi)) + \phi[\gamma(U) + \alpha \pi - p]
\]

The optimum conditions would then be
\[
\frac{\partial L}{\partial p} = \frac{\partial v}{\partial p} e^\pi + \lambda b - \phi = 0 \\
\frac{\partial L}{\partial U} = \frac{\partial v}{\partial U} e^\pi + \phi \gamma' = 0 \\
\frac{\partial L}{\partial \phi} = \gamma(U) + \alpha \pi - p = 0 \\
-\dot{\lambda} = \lambda(-b) + \phi \alpha \\
\dot{\pi} = b(p - \pi)
\]

If we specify a functional form (see Chiang chapter 7) we can find that the optimal path for policy, which shows that the political process creates a business cycle. To get the solution, it is often easier to find the solution by using equality constraints to eliminate variables before getting started. However, it is also true that there is economic meaning in the shadow prices, so your analysis can be enriched by solving the problem with the constraints stated explicitly.

II. Optimal control with inequality constraints

A. Theory

Suppose now that the problem we face is one in which we have inequality constraints, \( h_i(t, x, z) \leq c_i \), with \( i = 1, \ldots, n \), for \( n \) constraints and \( x \) and \( z \) are assumed to be vectors of the state and control variables respectively. For each \( x_j \in x \), the state equation takes the form \( \dot{x}_j = g_j(t, x, z) \).

As with standard constrained optimization problems, the Kuhn-Tucker conditions will yield a global maximum if any one of the Arrow-Hurwicz-Uzawa constraint
qualifications is met (see Chiang p. 278). The way this is typically satisfied in most economic problems is for the $h_j$ to be concave or linear in the control variables.

Assuming that the constraint qualification is met, we can then proceed to use the Lagrangian specification using a Hamiltonian which takes the form

$$H(t,x,z,\lambda) = u(t,x,z) + \sum_{j=1}^{m} \lambda_j g_j(t,x,z)$$

which we then plug into the Lagrangian with the constraints,

$$L = H(t,x,z,\lambda) + \sum_{i=1}^{n} \phi_i (c_i - h_i(t,x,z))$$

$$L = u(t,x,z) + \sum_{j=1}^{m} \lambda_j g_j(t,x,z) + \sum_{i=1}^{n} \phi_i (c_i - h_i(t,x,z))$$

Note: For maximization problems I always write the constraint term of the Lagrangian so that the argument inside the parentheses is constrained to be greater than zero, or for minimization problems you write it so that the argument is less than zero. If you follow this rule, your Lagrange multiplier will always be greater than or equal to zero.

The FOC’s for this problem are:

$$\frac{\partial L}{\partial z_k} = 0 \Rightarrow \frac{\partial u}{\partial z_k} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial z_k} - \sum_{i=1}^{n} \phi_i \frac{\partial h_i}{\partial z_k} = 0 \quad \text{for all } z_k \in z$$

$$\frac{\partial L}{\partial x_j} = -\dot{\lambda}_{ij} \quad \text{for all } j$$

$$\frac{\partial L}{\partial \lambda_j} = \dot{x}_j$$

and, for the constraints

$$\frac{\partial L}{\partial \phi_i} \geq 0 \Rightarrow h_i(x,z) \leq c_i$$

with the complementary slackness conditions:

$$\phi_i \geq 0 \text{ and } \phi_i \frac{\partial L}{\partial \phi_i} = 0 \text{ for all } i.$$  

As with all such problems, the appropriate transversality conditions must be used and, if you choose to use a current-value Hamiltonian, the necessary adjustments must be made. Note that in the current value specification, the interpretation of both the co-state variable and the shadow price on the intratemporal constraint would be altered.

Chiang solves the problem for a specific functional form and finds that if politicians solve an optimal policy path like this, unemployment will tend to fall as an election approaches and then rise again immediately after the election.
B.  Example: Hotelling’s optimal extraction problem

We return to Hotelling’s problem from Lecture 6. The planner’s problem is to maximize

\[ \max_{z_T} \int_0^T e^{-rt} \left[ \int_0^z p(z) \, dz \right] \, dt \quad \text{s.t.} \]

\[ \dot{x} = -z \]

\[ x(0) = x_0, \quad x_t \geq 0. \]

Economic intuition tells us that \( x_T = 0 \). However, we found in lecture 6 that it is possible to find a solution in which \( x_t \) becomes negative and then, \( z_t \) is negative for a period to restore \( x_t \) so that \( x_T = 0 \). However, by constraining \( x_T = 0 \) and \( z_t \geq 0 \) for all \( t \), we can indirectly ensure that \( x_t \geq 0 \) for all \( t \). The associated Lagrangian would then be

\[ L = e^{-rt} u(\cdot) + \lambda (-z_t) + \phi_t z_t. \]

We cover constraints on the state variable below

The associated maximization criteria are:

3. \( L_z = 0: \quad e^{-rt} u'(\cdot) - \lambda_t + \phi_t = 0 \Rightarrow e^{-rt} p(z_t) - \lambda_t + \phi_t = 0 \)

4. \( L_t = -\dot{\lambda}_t: \quad -\dot{\lambda}_t = 0 \)

5. \( L_x = 0: \quad \dot{x}_t = -z_t \)

6. \( L_z \geq 0: \quad z_t \geq 0 \)

7. \( \phi_t \geq 0 \)

8. \( \phi_t z_t = 0 \)

Kuhn Tucker Conditions

The transversality condition is \( x_T = 0 \).

From 4 it still holds that \( \lambda \) is constant as we found in Lecture 6. However, 3 can be rewritten

\[ p(z_t) = (\lambda_t - \phi_t)e^{rt}. \]

Using the assumed functional form for inverse demand curve, \( p(z) = e^{-\gamma x} \), we obtain

\[ e^{-\gamma z_t} = (\lambda_t - \phi_t)e^{rt} \]. Taking logs we get

\[ -\gamma z_t = \ln(\lambda_t - \phi_t) + rt \], or

9. \( z_t = -\frac{\ln(\lambda_t - \phi_t) + rt}{\gamma} \).

Now, using the complementary slackness conditions, we know that if \( z > 0 \) then \( \phi = 0 \) and if \( z = 0 \), \( \phi > 0 \). The path can, therefore, be broken into two parts, the first part from 0 to \( T_1 \) during which \( z > 0 \) and the second part, from \( T_1 \) to \( T \), where \( z = 0 \) and \( \phi > 0 \).

From 0 to \( T_1 \),

\[ z = -\frac{\ln(\lambda - 0) + rt}{\gamma} = -\frac{\ln(\lambda) + rt}{\gamma} > 0 \]

so, from the complementary slackness condition, 8, \( \phi = 0 \).

And from \( T_1 \) to \( T \),

\[ 0 = -\frac{\ln(\lambda - \phi_t) + rt}{\gamma} \Rightarrow \ln(\lambda - \phi_t) = -rt \], so that

10. \( \phi_t = \lambda_t - e^{-rt} \).
Hence, $\phi_{T_1}$ is increasing since the second term, $e^{-rt}$, gets smaller as $t$ increases.

Now, we can speculate about the solution. A critical question is whether $z$ and $\phi$ will be discontinuous over the planning horizon; i.e. will $z$ approach 0 at $T_1$ gradually, reaching 0 only at $T_1$, or will it jump from a positive level to 0 at the instant that it reaches $T_1$?

Imagine two possible paths. In the first path the level of consumption is at a positive level, $2 \cdot \epsilon$ for the period $\Delta$ prior to $T_1$ and zero from $T_1$ onward. In the second path consumption is at $\epsilon$ for $\Delta$ prior to $T_1$ and $\Delta$ afterward. The total amount of consumption is the same, $2 \cdot \epsilon$, but, by Jensen’s inequality we know that the utility over the short increment of time, $2 \Delta$, will be greater in the second case. Hence, any time there is a discontinuous jump in utility, we know that there is a preferred path with half the jump. Only a path in which consumption decreases continuously to zero does not violate this principal; a discontinuous jump will never be optimal.

So we can start by assuming that $z_t$ approaches 0 continuously as $t \rightarrow T_1$. Under this assumption, $\phi_t = 0$ for $t \leq T_1$ so that, from 10,

$$\lambda_{T_1} = e^{-rT_1}.$$  

Furthermore, we know that since $z_t = 0$ from $T_1$ onward, we must exhaust the resource by $T_1$:

$$\int_0^{T_1} z_t dt = x_0 \text{ or } \int_0^{T_1} \left(-\frac{\ln(\lambda) + rt}{\gamma}\right) dt = x_0$$  

Which we solved in lecture 6 to obtain

$$\lambda = \left(\frac{x_0 - x_{T_1}}{z_{T_1}}\right).$$  

Combining 11 and 13, we obtain

$$e^{-rT_1} = e^{\left(-\frac{x_0 - x_{T_1}}{z_{T_1}}\right)}} \Rightarrow \frac{r}{2} T_1 = \frac{\gamma}{T_1} x_0 \Rightarrow T_1^2 = \frac{2\gamma}{r} x_0,$$  

which can be simplified to

$$T_1 = \sqrt{\frac{2\gamma}{r} x_0}.$$  

Hence, the resource will be exhausted by $T_1$ and the constraint on $z$ is binding from $T_1$ onwards. Finally, for $t > T_1$, recall that $\phi_t = \lambda - e^{-rt}$, so $\phi_t = e^{\left(-\frac{x_0 - x_{T_1}}{z_{T_1}}\right)} - e^{-rt}$. This is the shadow price of the constraint on $z$, which gets larger for $t > T_1$.

III. Constraints on the state space

A. Theory

Suppose now that we have constraints on the state variables which define a feasible range. This is common in economic problems. You may, for example, have limited storage space so that you cannot accumulate your inventory forever. Or, if you were dealing with a biological problem, you might be constrained to keep your stock of a
species above a lower bound where reproduction begins to fail, and an upper bound where epidemics are common.

The approach to such problems is similar to that of the control problems. Suppose we have an objective function

\[
\max \int_0^T u(t, x, z)dt \text{ s.t.} \\
x = g(t, x, z), \ x(0) = x_0 \text{ and} \\
h(t, x) \geq 0.
\]

The augmented Hamiltonian for this problem is

\[
L = u(t, x, z) + \lambda g(t, x, z) + \phi h(t, x)
\]

and the necessary conditions for optimality include, the constraints plus

\[
\frac{\partial L}{\partial z} = 0 \\
\hat{\lambda} = -\frac{\partial L}{\partial x} \\
\phi \geq 0 \text{ and } \phi h = 0
\]

and the transversality condition.

Solving problems like this by hand can be quite difficult, even for very simple problems. (See K&S p.232 if you want to convince yourself). (An alternative approach presented in Chiang (p. 300) is often easier and we follow this approach below). For much applied analysis, however, there may be no alternative to setting a computer to the problem to find a numerical solution.

B. Example: Hotelling’s optimal extraction problem

Clearly, Hotelling’s problem can also be modeled as a restriction that \( x_i \geq 0 \). In this case our Lagrangian would take the form

\[
L = e^{-rt}u(\cdot) + \lambda(z_i) + \phi_t x_t.
\]

And the associated maximization criteria are:

15. \( L_z = 0 \):
\[
e^{-rt} u'(\cdot) - \lambda_t = 0 \Rightarrow e^{-rt} p(z_t) - \lambda_t = 0
\]

16. \( L_{\lambda} = -\lambda_t \):
\[
-\dot{\lambda}_t = \phi_t
\]

17. \( L_{\dot{x}_t} = \dot{x}_t = -z_t \)

18. \( L_{x_t} \geq 0 \):
\[
x_t \geq 0
\]

19. \( \phi \geq 0 \)

20. \( \phi x_t = 0 \) \{ Kuhn Tucker Conditions \}

We won’t solve this problem in all its detail, but the solution method would follow a similar path to that used above. We divide time into two portions, from 0 to \( T_1 \) where \( \phi = 0 \) and \( \lambda \) is constant, and from \( T_1 \) to \( T \), where \( x_t = 0 \) and \( \lambda \) falls with and \( \phi \) increases.

To solve the problem, we use the same logic as above to determine that \( \phi T_1 = 0 \). We can then solve \( T_1 \) to obtain 14.
One thing that is interesting in this specification is that the co-state variable is no longer constant over time. This makes sense since between 0 and $T_1$ we are indifferent about when we get the extra unit of the resource. But after $T_1$, it clearly makes a difference—the sooner we could obtain a marginal unit the more valuable (in PV terms) it will be. When $t > T_1$, we know that $z_t = 0 \Rightarrow p = 1$ (the choke price) and $\lambda_t = e^{-rt}$. A marginal increase in the stock over this range would allow the immediate sale of that stock at a price of 1 and the present value of this marginal change in stock would, therefore, be $e^{-rt}$.

The economic meaning of $\phi_t$ is also of interest. From 16, $-\dot{\lambda} = \phi_t$. This means that the shadow price on the inequality constraint is inversely related to the rate of change in the shadow price in the co-state variable. For the period when $\lambda$ is constant, there’s no shadow price on the state variable—there’s no shortage yet. As $\lambda$ starts to decline, $\phi$ becomes positive, and

$$\int_{T_1}^{T} -\dot{\lambda} dt = \lambda_{T_1} - \lambda_T = \int_{T_1}^{T} \phi_t dt$$

So $\phi_t$ can be thought of as the instantaneous cost of the inequality constraint, while $\lambda_t$ is the accumulation of those instantaneous costs.

IV. Bang-bang OC problems

We now consider problems for which the optimal path does not involve a smooth approach to the steady state or gradual changes over time. Two important classes of such problems are known as "bang-bang" problems and most rapid approach problems. In such problems the constraints play a central role in the solution.

A. Bang-bang example #1: A state variable constraint

Consider the following problem in which we seek to maximize discounted linear utility obtained from a nonrenewable stock (sometimes referred to as a cake-eating problem):

$$\max_{z} \int_{0}^{T} e^{-rt} z_t dt \quad \text{s.t.}$$

$$\dot{x} = -z$$

$$x(t) \geq 0$$

$$x(0) = x_0$$

What does intuition suggest about the solution to the problem? Will we want to consume the resource stock $x$ gradually? Why or why not? Let’s check our intuition.

Following the framework from above, we set up the Lagrangian by adding the constraint on the state variable to the Hamiltonian, i.e., $L = H + \phi$ (constraint). Using the current-value specification, this gives us

$$L = z_t - \mu_t z_t + \phi_t x_t$$

The FOCs for the problem are:

(i) $\frac{\partial L}{\partial z} = 0 : \quad 1 - \mu_t = 0$
(ii) \( \frac{\partial L}{\partial x} = r\mu_i - \dot{\mu}_t : \quad \phi_t = r\mu_i - \dot{\mu}_t \)

Because of the constraint, the complementary slackness condition must also hold:
(iii) \( \phi_t x_t = 0 \).

Equation i implies that \( \mu = 1 \). Since this holds no matter the value of \( t \), we know that \( \dot{\mu}_t = 0 \) for all \( t \). Conditions i and ii together indicate that \( \mu = 1 \) and \( \phi = r \).

The second of these is most interesting. It shows us that \( \phi_t \), the Lagrange multiplier, is always positive. From the complementary slackness condition, it follows that \( x_t \) must equal 0 always. But wait! We know this isn't actually true at \( t=0 \); but, at \( t=0 \), \( x_t \) is not variable – it is parametric to our problem. Since \( x_0 \) cannot be chosen, this condition only applies for \( t>0 \); at every instant except the immediate starting value, \( x_t=0 \).

So how big is \( z \) at zero? The first thought is that it must equal \( x_0 \) but this isn't quite right. To see this, suppose that we found that the constraint started to bind, not immediately, but after 10 seconds. To get the \( x \) to zero in 10 seconds, \( z \) per second would have to equal \( x_0/10 \). Now take the limit of this at the denominator goes to zero \( \Rightarrow z \) goes to infinity. Hence, what happens is that for one instant there is a spike of \( z_t \) of infinite height and zero length that pushes \( x \) exactly to zero. This type of solution is known as a bang-bang problem because the state variable jumps discontinuously at a single point – BANG-BANG! Since, in the real world it's pretty difficult to push anything to infinity, we would typically interpret this solution as “consume it as fast as you can.” This is formalized in the framework of most-rapid-approach path problems below.

B. Bang-Bang Example #2 (based on Kamien and Schwartz p. 205) A control variable constraint

Let \( x_t \) be a productive asset that generates output at the rate \( r x_t \). This output can either be consumed or reinvested. The portion that is reinvested will be called \( z_t \) so \( [1-z_t] \) is the portion that is consumed. We assume that the interest can be consumed, but the principal cannot be touched.\(^3\) Our question is, What portion of the interest should be invested and what portion should be consumed over the interval \([0,T]\)?

Formally, the problem is:
\[
\max_z \int_0^T [1-z_t] r x_t dt \quad \text{s.t.} \quad \dot{x}_t = z_t r x_t, \quad 0 \leq z_t \leq 1, \quad x(0) = x_0
\]

This time we have two constraints: \( z \leq 1 \) and \( z \geq 0 \). Hence, our Lagrangian is
\[
L = [1-z_t] r x_t + \lambda z_t r x_t + \phi_t (1-z_t) + \phi_z z_t,
\]

\(^3\)This problem is very similar to one looked at in Lecture 3. Comparing the two you’ll see one key difference is that here utility is linear, while in lecture 3 utility was logarithmic.
where \([1-z_t]rx_t + \lambda z_t r_x\) is the Hamiltonian part of the problem and the last two terms are the constraints.

The necessary conditions for an optimum are

1. \[
\frac{\partial L}{\partial z} = 0 \iff -rx_t + \lambda r_x - \phi_1 + \phi_2 = 0,
\]
2. \[
\frac{\partial L}{\partial \lambda} = -\dot{\lambda} \iff -\dot{\lambda} = [1-z_t]r + \lambda z_t r_x.
\]

The transversality condition in this problem is \(\lambda_T = 0\) since \(x_T\) is unconstrained with the Kuhn-Tucker conditions,

\(KT_1\): \(\phi_1 \geq 0\ \&\ \phi_1(1-z) = 0\), and
\(KT_2\): \(\phi_2 \geq 0\ \&\ \phi_2 z = 0\).

From the \(KT_1\), we know that if \(\phi_1 > 0\), then the first constraint binds and \(z = 1\). Similarly, from \(KT_2\), if \(\phi_2 > 0\), then the second constraint binds and \(z = 0\). i.e.

\[
\begin{align*}
\phi_1 > 0 \implies z = 1 & \quad \phi_2 > 0 \implies z = 0. \\
\phi_1 = 0 \implies z < 1 & \quad \phi_2 = 0 \implies z > 0.
\end{align*}
\]

Clearly, it is not possible for both \(\phi_1\) and \(\phi_2\) to be positive at the same time.

The first FOC can be rewritten

\[
(\lambda_t - 1)rx_t - \phi_1 + \phi_2 = 0.
\]

We know that \(rx_t\) will always be positive since consumption of the capital stock is not allowed. Hence, we can see that three cases are possible:

1) if \(\lambda_t = 1\) \(\implies \phi_1 = 0\ \&\ \phi_2 = 0\ \implies\) no constraint binds
2) if \(\lambda_t > 1\) \(\implies \phi_1 > 0\ \&\ \phi_2 = 0\ \implies z = 1\)
3) if \(\lambda_t < 1\) \(\implies \phi_1 = 0\ \&\ \phi_2 > 0\ \implies z = 0\).

From the second FOC,

\[
\dot{\lambda} = -(1 - z_t) r + \lambda z_t r_x.
\]

Since everything in the brackets is positive, the RHS of the equation is negative \(\implies \lambda\) is always falling.

By the transversality condition we know that eventually \(\lambda\) must hit \(\lambda_T = 0\). Hence, eventually we'll reach case 3 where, \(\lambda_t < 1\) and \(z = 0\) and we consume all of our output. But when do we start consuming, right away or after \(x\) has grown for a while? We know from equation 2 that at \(\lambda_t = 1\) neither constraint binds.

- Suppose that at \(t=n\) \(\lambda_t = 1\).
- For \(t<n\) \(\lambda_t > 1\) and \(z = 1\).
- For \(t>n\) \(\lambda_t < 1\) and \(z = 0\).
An important question then is when is \( n \)? We can figure this out by working backwards from \( \lambda_t=0 \). From the second FOC, we know that in the final period, (when \( \lambda_t<1 \)) \( z=0 \), in which case
\[
\dot{\lambda} = -r.
\]
Solving this differential equation yields
\[
\lambda_t = -rt + A.
\]
Using the transversality condition,
\[
\lambda_r = -rT + A = 0
\]
\[A = rT \]
\[
\lambda_t = -rt + rT = r(T-t)
\]
Hence, \( \lambda_t=1 \) if
\[
r(T-n) = 1
\]
\[
n = (rT-1)/r
\]
Hence, we find that the optimal strategy is to invest everything from \( t=0 \) until 
\[
t = n = (rT-1)/r.
\]
After \( t=n \) consume all of the interest. If \( (rT-1)/r < 0 \) then it would be optimal to consume everything from the very outset.

For \( (rT-1)/r > 0 \), we can graph the solution:

What would be the solution as \( T \to \infty \)? Does this make intuitive sense? What is it about the specification of the problem that makes it inconsistent with our economic intuition?

V. Most Rapid Approach Path problems

Bang-bang problems fit into a general class of problems that are commonly found in economics: most-rapid-approach path problems (MRAP).\(^4\) Here, the optimal policy is to get as quickly as possible to steady state where benefits are maximized. Consider the first example bang-bang example above. Wouldn’t a solution in which we move toward the equilibrium as fast as possible rather than impossibly fast be more intuitively appealing?

\(^4\) Sometimes the term “bang-bang” is also used to describe MRAP problems.
A. **MRAP example (Kamien & Schwartz p. 211)**

A very simple firm generates output from its capital stock with the function $f(x_t)$ with the property that

$$\lim_{x \to 0} f'(x) = \infty.$$ 

The profit rate, therefore, is

$$\pi_t = p \cdot f(x_t) - c \cdot z_t$$

where $x_t$ is the firm's capital stock and $z_t$ is investment, $p_t$ and $c_t$ are exogenously evolving unit price and unit cost respectively. The capital stock that starts with $x(0)=x_0$, depreciates at the rate $b$ so that

$$\dot{x}_t = z_t - bx_t.$$ 

The firm's problem, therefore, is to maximize the present value of its profits,

$$\int_0^\infty e^{-rt} \left[ p \cdot f(x_t) - c \cdot z_t \right] dt$$ 

subject to

$$\dot{x}_t = z_t - bx_t,$$

with three additional constraints:

i) $x(t) \geq 0$

ii) $z_t \geq 0$

iii) $p \cdot f(x_t) - c \cdot z_t \geq 0$

Let's use economic intuition to help us decide if we need to explicitly include all the constraints in solving the problem?

- The constraint on $x$ almost certainly does not need to be imposed because as long as $f'$ gets big as $x \to 0$, the optimal solution will always avoid zero.

- The constraints on $z$, on the other hand might be relevant. But, we'll start by assuming that neither constraint binds, and then see if we can figure out actual the solution based on the assumed interior solution or, if not, we'll need to use the Kuhn-Tucker specification. Note that if there does exist a steady state in $x_t$, then, as long as $b>0$, $z_t$ must be greater than zero. Hence, we anticipate that much might be learned from the interior solution.

- Similarly, the profit constraint might also bind, but we would expect that in the long run, profits would be positive. So again, we start by solving for an interior solution, assuming $\pi > 0$ where $\pi = p \cdot f(x_t) - c \cdot z_t$.

B. **The interior solution**

The current value Hamiltonian of the problem (assuming an interior solution w.r.t. $z$ and $x$ with $\pi > 0$) is

$$H_c = p \cdot f(x_t) - c \cdot z_t + \mu_t (z_t - bx_t)$$

The necessary conditions for an interior solution are:

$$\frac{\partial H_c}{\partial z_t} = 0 \quad \Rightarrow \quad -c + \mu_t = 0$$

$$\frac{\partial H_c}{\partial x_t} = r \mu_t - \mu_t \Rightarrow \quad p \frac{df(x_t)}{dx_t} - \mu_t b = r \mu_t - \mu_t$$

Over any range where the constraints on $z$ do not bind, therefore, we have

$c = \mu_t$ 

and, therefore, it must also hold that
\[ \ddot{\mu} = \dot{c} = 0. \]

Substituting \( c \) for \( \mu \) and rearranging, the second FOC becomes

\[ 23. \quad p_i \frac{\partial f(x)}{\partial x_i} = (r + b) c, \]

which must hold over any interval where \( z > 0 \), i.e. the constraint is not binding.

We see, therefore, that the optimum conditions tell us about the optimal level of \( x \), say \( x^* \). We can then use the state equation to find the value of \( z \) that maintains this relation.

Since \( c \) and \( p \) are constant, this means that the capital stock will be held at a constant level and 23 reduces to \( \frac{pf'(x)}{r + b} = c \). This is known as the modified golden rule.

Let's think about this condition for a moment.

- In a static optimization problem, the optimal choice would be to choose \( x \) where the marginal product of increasing \( x \) is equal to the marginal cost, i.e., where \( pf' = c \).
- In an infinite-horizon economy, if we could increase \( x \) at all points in time this would have a discounted present value of \( \frac{pf'}{r} \). However, since the capital stock depreciates over time, this depreciation rate diminishes the present value of the gains that can be obtained from an increase in \( x \) today, hence the present value of the benefit of a marginal increase in \( x \) is \( \frac{pf'}{r + b} \).

If \( p \) and \( c \) are not constant, but grow in a deterministic way (e.g., constant and equal inflation) then we could de-trend the values and find a real steady state. If \( p \) and \( c \) both grow at a constant rate, say \( w \), then there will be a unique and steady optimal value of \( x \) for all \( z > 0 \).

Hence, our first-order conditions can be used to learn a lot about the nature of the situation when the constraints do not bind, i.e. when \( z > 0 \) and \( p \cdot f(x) > c \cdot z \). However, this is not the end of the story.

C. Corner solutions

All of the discussion above assumed that we are at an interior solution, where \( 0 < z_t < p \cdot f(x_t)/c \). But, we found that the interior solution only holds when the state variable \( x \) is at the point defined by equation 23; if the value of \( x \) is not at \( x^* \) at \( t=0 \), then it must be that we have a corner solution in which either \( z_t = 0 \) or \( p \cdot f(x_t) - c \cdot z_t = 0 \).

If \( x_0 > x^* \) then it will follow that \( z \) will equal zero until \( x_t \) depreciates to \( x^* \). If \( x_0 < x^* \) then \( z \) will be as large as possible \( \frac{p}{c} f(x_t) = z_t \) until \( x^* \) is reached.
Hence, economic intuition and a good understanding of the steady state can tell us where we want to get and how we're going to get there – in the most rapid approach possible.

D. Some theory and generalities regarding MRAP problems

A general statement of the conditions required for a MRAP result is presented by Wilen (1985, p. 64):

Spence and Starrett show that for any problem whose augmented integrand (derived by substituting the dynamic constraint into the original integrand) can be written as

$$\Pi_A(K, \dot{K}) = M(K) + N(K)\dot{K}$$

the optimal solution reduces to one of simply reaching a steady state level $K=K^*$ as quickly as possible.

Where $K$ is the state variable and by "integrand" they mean the objective function, profits in the case considered here.

How does this rule apply here? The integrand is $p_t f(x_t) - c_t z_t$. Using the state equation $bx_t + \dot{x}_t = z_t$, the integrand can be written

$$p_t f(x_t) - c_t (bx_t + \dot{x}_t) = p_t f(x_t) - c_t bx_t - c_t \dot{x}_t.$$  

Converting this to the notation used by Wilen,

$$M(K) = p_t f(x_t) - c_t bx_t$$

and

$$N(K)\dot{K} = c_t \dot{x}_t.$$  

Hence this problem fits into the general class of MRAP problems.

For a more intuitive understanding of why bang-bang and MRAP solutions arise, consider the general problem of the form

$$\max_{z_t} \int_0^T e^{-\mu t} f(x_t) z_t dt \quad \text{s.t.} \quad x_t = g(x_t) z_t$$

so that both the benefit function and the state equation are linear in $z_t$. In this case, the Hamiltonian would be written

$$H_c = f(x_t) z_t + \mu_t \cdot g(x_t) z_t.$$  

The optimization criterion remains: Maximize $H_c$ with respect to $z_t$ for all $t$. If $f(x_t) + \mu_t \cdot g(x_t) > 0$, then to maximize $H_c$ we should set $z_t$ at $+\infty$. If $f(x_t) + \mu_t \cdot g(x_t) < 0$, then $z_t$ should be set at $-\infty$. Hence, both the benefit function and the state equation are linear in $z$ a bang-bang or MRAP solution will be obtained.

One lesson that can be obtained from this is that you need to be careful when specifying your model. While linear functions are nice to work with and frequently offer nice

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5 I found this simple presentation in Rodriguez et al. (2011), though I imagine that the presentation has been presented by others previously.
intuition, they can also lead to corner solutions that are not intuitive, may not be easy to work with and may lack the intuitive economic meaning that the model is set up to deliver.

VI. References


