Utility function, Risk Aversion and Option Pricing

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• Recall that European call options are priced as

\[
C(X; \tau) = \exp(-r\tau) \int_x^\infty (S_T - X) f(S_T) dS_T \\
= \exp(-r\tau) \int_x^\infty (S_T - X) f(S_T) p(S_T) dS_T \\
= \exp(-r\tau) \int_x^\infty (S_T - X) K(S_T) p(S_T) dS_T 
\]

where \( X \) is the strike price, \( f \) is the risk neutral density, \( p \) is the physical or subjective density, and \( K \) is the pricing kernel.

• The RND \( f \) is determined by the no arbitrage condition. Consider a two period investment with one stock and one bond, whose interest rate is \( r = 0 \). The current price of stock is $250, and its price for the next period is expected to take two possible values $400 and $200 with equal possibilities. How do we price an European call option with a strike of $300?

Given that the expected stock price is $300 and the current price is $250, does it make sense to price it at $50? Suppose the option is indeed priced at $50. A seller at period one can sell this option and borrow $200 amount of cash (bond) to buy one unit of stock. At the second period, if stock price is $400, the option will be exercised. The net payoff is $300 − $200 = $100. If the stock price is $200, the option will not be exercised. The net payoff is $200 − $200 = $0. In either way, the seller is guaranteed to make non-negative profit. This is arbitrage.

The complete market hypothesis rules out the possibility of arbitrage. To get to the no-arbitrage price of the option. Suppose a seller can hold \( y \) unit of stock and \( x \) unit of bond as a portfolio. In the second period, the seller needs \( $(S_T - 300)_+ \) in order to meet the claim against her. In other words, to avoid negative profits, she needs

\[
x + 400y \geq (400 - 300)_+ = 100, \quad x + 200y \geq (200 - 300)_+ = 0.
\]
Solving for two equations yields $x = -100$ and $y = 1/2$, under which the claims are exactly replicated. The cost of building this portfolio at period one is $250/2 - 100 = 25$. Hence the price of this option is $25$.

Note that the solution is invariant to the probabilities of the second stage stock price. It is determined by the risk free interest rate (zero in the current case for simplicity), hence the name risk-free density.

- The discrepancy between the physical and risk neutral densities reflects agents’ risk attitude. The following exposition borrows heavily on Jackwerth (2000). Consider a representative agent in a complete market. She is endowed with one unit of wealth and a fixed time horizon $t$. The problem is to maximize utility across future wealth states subject to the budget constraint

$$\max_W \int p(W)U(W)dW - \lambda \left( \frac{1}{r^t} \int f(W)WdW - 1 \right)$$

where $W$ is future wealth, $U$ is a state-independent utility function, $\lambda$ is the shadow price of budget constraint, $r$ is the risk free return rate.

Let $S$ be the return of market portfolio across states. In equilibrium, it satisfies the first order condition

$$U'(S) = \frac{\lambda f(S)}{r^t p(S)}$$

Since all RHS quantities are positive, it implies positive marginal utility. However, since the shadow price is unknown and can’t be easily estimated, it is difficult to infer the implied utility function using this identity.

Further differentiating the first order condition gives

$$U''(S) = \frac{\lambda f'(S)p(S) - f(S)p'(S)}{r^t p^2(S)}$$

It follows that

$$-\frac{U''(S)}{U'(S)} = \frac{p'(S)}{p(S)} - \frac{f'(S)}{f(S)} = -\frac{\partial \log K(S)}{\partial S}$$

Note the LHS is the absolute risk aversion. Thus it is seen that the pricing kernel is linked
directly to agents’ risk aversion.

- A large bodies of empirical work have documented non-decreasing risk aversion and some have attempted to provide answers to this abnormality. E.g., state-dependence (in belief or utility) has been suggested to solve this puzzle (Brown and Jackwerth (2004), Chabi-Yo et al. (2008)). Polkovnichenko and Zhao (2013) explored probability weighting due to rank dependent expected utility. Yet another possible explanation is habit formation, which relaxes the assumption of inter-temporal separability of utility.

There also exist some work that use more sophisticated methods in the estimation of pricing kernel, which does not support the abnormality in risk aversion. See, e.g., Chaudhuri and Schroder (2015) and references therein.

References


