Black Scholes Model

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• Suppose a stock return between two points in time follows a normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Partition the time interval into \( n \) periods such that \( \Delta t = 1/n \). Let the return for the \( i \)-th sub-period be \( r_i \) and \( r = \sum_{i=1}^{n} r_i \) and \( r_i \)'s are assumed to be i.i.d.. We then have \( \mu = E[\sum_{i=1}^{n} r_i] = nE[r_i] \rightarrow E[r_i] = \mu \Delta t \), and \( \sigma^2 = Var(\sum_{i=1}^{n} r_i) = nVar(r_i) \rightarrow Var(r_i) = \sigma^2 \Delta t \). This is a simple random walk model, which implies that the mean and variance of increment are both of order \( O(\Delta t) \).

• The simple random walk model above can be viewed as a discretization of Brownian motion, or Wiener process. The Wiener process, \( W_t \), is a stochastic process characterized by

1. \( W_0 = 0 \)
2. The function \( t \rightarrow W_t \) is almost surely everywhere continuous
3. \( W_t \) has independent increments such that \( W_t - W_s \sim N(0, t-s) \), \( 0 \leq s < t \)

Some properties

1. \( f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} exp(-\frac{x^2}{2t}) \)
2. \( E(W_t) = 0 \)
3. \( Var(W_t) = t \)
4. \( Cov(W_t, W_s) = \min(s, t) \)

Suppose \( s < t \). The last property is given by \( Cov(W_t, W_s) = E[W_tW_s] - E[W_t]E[W_s] = E[(W_s + W_t - W_s)W_s] = E[W_t^2] + E[W_t - W_s]E[W_s] = Var(W_s) = s \).

• Itô’s Lemma. Let \( X_t \) be an Itô’s diffusion process governed by

\[
dX_t = \mu_t dt + \sigma_t dW_t
\]
where \( W_t \) is a Wiener process. Note \( \int_0^T dW_t = N(0, T) \) is a normal random variable with variance \( T \). This is a stochastic integration.

Suppose \( f(t, x) \) be a twice differentiable scalar function. Its Taylor's expansion is given by

\[
df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2 + \cdots
\]

Let \( x \) in \( f(t, x) \) be Itô's diffusion. We have

\[
df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\mu_t^2 dt^2 + 2\mu_t \sigma_t dtdW_t + \sigma_t^2 dW_t^2) + \cdots
\]

Suppose \( dt \to 0 \), then \( dt^2 \) and \( dtdW_t \to 0 \) faster than \( dW_t^2 \), which is of order \( O(dt) \). Write \( dW_t^2 \) as \( dt \), we then have

\[
df = \left( \frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t
\]

- Suppose stock return follows a diffusion process \( \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \), with \( dW_t \sim N(0, dt) \). Let \( f = f(S_t) \). Applying Itô’s Lemma gives

\[
df = (\mu S_t \frac{df}{dS} + \frac{1}{2} \sigma^2 S_t^2 \frac{d^2 f}{dS^2}) dt + \sigma S_t \frac{df}{dS} dW_t
\]

Let \( f(S_t) = \ln S_t \), we have \( f'(S) = 1/S \) and \( f''(S) = -1/S^2 \). It follows that

\[
df_t = (\mu - \sigma^2/2) dt + \sigma dW_t
\]

We then have

\[
\int_0^T df_t dt = \int_0^T \ln(S_t) dt = \int_0^T (\mu - \sigma^2/2) dt + \sigma \int_0^T dW_t
\]

Thus \( \ln(S_T) - \ln(S_0) = N((\mu - \sigma^2/2)T, \sigma^2 T) \) and \( \ln(S_T) \sim N(\ln(S_0) + (\mu - \sigma^2/2)T, \sigma^2 T) \).

- Let \( C(t, S) \) be the value of an option on a stock with price \( S \). Assume that \( dS = \mu S dt + \sigma S dW_t \). We then have

\[
dC = \left( \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW_t
\]

Consider a delta-hedge portfolio, which is short one option and long \( \frac{\partial C}{\partial S} \) of the underlying
stock. Its value is given by

\[ V = -C + \frac{\partial C}{\partial S} S \]

Suppose in a short time interval \([t, t + \Delta t]\),

\[ \Delta V = -\Delta C + \frac{\partial C}{\partial S} \Delta S \]

and

\[ \Delta S = \mu S \Delta t + \sigma S \Delta W \]
\[ \Delta C = (\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}) \Delta t + \sigma S \frac{\partial C}{\partial S} \Delta W \]

It follows that

\[ \Delta V = (-\frac{\partial C}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}) \Delta t \]

Further assume there is a risk free interest rate \(r\) such that under the no-arbitrage condition

\[ r V \Delta t = \Delta V \]

We then have

\[ r(-C + \frac{\partial C}{\partial S} S) \Delta t = (-\frac{\partial C}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}) \Delta t \]

which gives the Black-Scholes equation

\[ rS \frac{\partial C}{\partial S} - rC + \frac{\partial C}{\partial t} - \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} = 0. \]

Now consider a European call option with strike \(K\). Solving the Black-Scholes PDE gives

\[ C(S_t, t) = S_t \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) \]

where

\[ d_1 = \frac{1}{\sigma \sqrt{T-t}} \left\{ \ln(S_t/K) + (r + \sigma^2/2)(T-t) \right\} \]

and

\[ d_2 = \frac{1}{\sigma \sqrt{T-t}} \left\{ \ln(S_t/K) + (r - \sigma^2/2)(T-t) \right\} = d_1 - \sigma \sqrt{T-t} \]
• Note that in the Black-Scholes option pricing formula, the first term is the current stock price adjusted by $\Phi(d_1)$, and the second term in the present value of strike adjusted by $\Phi(d_2)$. In particular, $\Phi(d_1)$ is what we refer to as option Delta, which is the hedge ratio of shares of stocks to options necessary to maintain a fully hedged position. The first term reflects the expected benefit of holding the option, and the value of the option is the difference between its expected benefit at time $t$ and the present value of the payoff at maturity.

Another Greek is the 'Vega', which is $\frac{\partial C}{\partial \sigma} = S_t \phi(d_1) \sqrt{T-t} > 0$. The larger is implied volatility, the more valuable an option is.

• Keep in mind that the Black Scholes model is derived under the following assumptions:
  
  – Efficient market
  – Known constant interest rate
  – No dividends nor transaction costs
  – European option
  – Normal stock return with known constant variance