THE THEORY OF MIXED DEMAND FUNCTIONS

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The properties of Samuelson's mixed demand functions, which express demand as a function of a mixed set of prices and quantities, are derived. By analyzing compensated (or substitution) effects and uncompensated effects, the relationships between mixed demand functions and conditional (or rationed) demands are examined. This provides insights on the behavioral implications of consumer theory for alternative demand specifications.

1. Introduction

Over the last few decades, consumption theory has had a profound impact on the investigation of consumer behavior. As illustrated in survey articles by Barten or Brown and Deaton, the economic theory of consumption is usually expressed in terms of demand functions relating quantities demanded to prices and consumer income. This is justified at the microlevel since, under perfect competition, quantities are then the choice variables while prices and income are parameters for a utility maximizing consumer.

However, these demand functions, expressing quantities as a function of prices, are not the only way, and not necessarily the best way, of specifying demand relationships. For example, using duality [Dievret (1974, 1982)], Hicks and Weymark have shown that inverse demand functions, expressing (normalized) prices as a function of quantities, are alternative ways of specifying the behavioral implications of consumer theory. Recently, inverse demand functions have been the subject of considerable interest in the literature [Anderson (1980), Christensen and Manser (1977), Laitinen and Theil (1979), Salves-Bronsard et al (1977)].

In between the quantity dependent demand functions and these inverse demands, Samuelson has argued that there exists a whole family of mixed demand functions that are also implied by consumer utility maximization. These mixed functions, which express demand relationships as a function of a mixed set of prices and quantities [Samuelson (1965, p. 791)], appear attractive since they provide added flexibility in the specification of demand

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functions. This flexibility may be important given the fact that most of the
customer demand studies based on time series data involve the estimation of
aggregate demand functions. The behavioral implications of consumer theory
are then typically assumed to hold at the aggregate level [Barten (1977)].
Since both prices and quantities are endogenous variables at the market
level, the rationality for using quantities dependent demand functions in this
context mostly disappears. Depending on the characteristics of each market,
one may want to specify some demand relationships as price dependent, and
others as quantity dependent [Hein (1977)]. Samuelson's mixed demand
functions would then be appropriate and would provide a theoretical basis
for studying aggregate consumption behavior. Although mixed demand
functions have been discussed in the context of production theory [Samuel-
son (1960), Bruno (1978)], the behavioral implications of consumer theory for
mixed demand functions are apparently not known. There is a need to derive
these implications in order to refine the theoretical properties of mixed
demand systems and make them operational for empirical investigation of
consumer behavior.

The primary objective of this paper is to do so. A secondary objective is to
present a more unified treatment of the behavioral implications of consumer
theory for alternative demand specifications. The specifications investigated
here include the more traditional 'pure' quantity dependent (or price
dependent) demands, the conditional or rationed demands [e.g., Pollak
(1969), Neary and Roberts (1980)] and Samuelson's mixed demands. After a
brief review of classical consumption theory in section 2, conditional demand
functions are discussed in section 3. The close relationship existing between
conditional demands and mixed demands will prove to be particularly useful
in the derivation of the properties of mixed demand functions. Such
properties are presented in section 4, where a number of well known duality
results of consumer theory are generalized. A discussion of a number of
connections among these alternative demand specifications is the topic of
section 5. In general, the relationships between compensated (or substitution)
effects and uncompensated effects play a crucial role throughout the paper.

2. Preliminaries

In this section, we briefly review some of the behavioral implications of
classical consumption theory. Such implications will prove useful in the
derivation of a number of results concerning mixed demand functions.

First, consider the following notation. Let \(x\) and \(p\) be positive vectors of
quantities and prices of \(n\) goods considered in the consumer's allocation
given an income level \(y\). Defining the normalized price vector \(v = p/y\), the
following functions play a crucial role in consumer theory [e.g., Weymark
(1986)].
the direct utility function $U(x)$, which is continuous, increasing and quasi-concave.

- the indirect utility function $V(v)$, which is continuous, decreasing and quasi-convex,

- the (normalized) expenditure function $E(U, v)$, which is continuous, increasing in $U$, linear homogeneous, increasing and concave in $v$,

- the transformation function $T(U, x)$, which is continuous, decreasing in $U$, linear homogeneous, increasing and concave in $x$.

Ignoring for the moment continuity problems at the boundaries of their domain of definition, these functions exhibit the following duality relationships [Dievert (1982), Blackorby et al. (1978), Weymark (1980), Anderson (1980)]:

(a) $V(v, 1) = \max_x U(x)$ subject to $v'x = 1$, (1)

where the solution $x(v, 1)$ is the vector of Marshallian demand functions,

(b) $U(x) = \min_v V(v, 1)$ subject to $v'x = 1$, (2)

where the solution $v(x, 1)$ is the vector of ordinary (normalized) inverse demand functions,

(c) $E(U^0, v) = \min_x v'x$ subject to $U^0 - U(x)$, (3)

where the solution $x^c(v, U^0)$ is the vector of compensated Hicksian demand functions,

(d) $E(V, v) = 1 \rightarrow V(v, 1)$, (4)

implying that the indirect utility function can be obtained by inverting the expenditure function,

(e) $T(U^0, x) = \min_v v'x$ subject to $U^0 = V(v)$, (5)

where the solution $v'(v, U^0)$ is the vector of compensated (normalized) inverse demand functions,

(f) $T(U, x) = 1 \rightarrow U(x)$, (6)

implying that the direct utility function can be obtained by inverting the
transformation function. These results indicate that it is possible to construct any one of the four functions $U$, $V$, $T$ and $E$ from any other function. Given the regularity conditions stated above, this duality always holds away from the boundaries of the domain of the functions. In order for these results to hold also at the boundaries, the functions must be extended in such a way as to preserve continuity. This extension has been discussed at length by Diewert (1974, 1982) and Weymark (1980). Given this extension, any one of the four functions $U$, $V$, $T$ and $E$ can represent consumer preferences. In particular, the specification of any one of the four functions is equivalent to specifying a preference ordering.

In this paper, we will restrict our attention to cases where the demand functions $x_0(v,1)$, $v(x,1)$, $x'(v, U^0)$ and $v'(x, U^0)$ are unique solutions of the optimization problems (1), (2), (3) and (5), respectively. This is done by strengthening the regularity conditions from convexity to strict convexity. For example, the direct utility function $U(x)$ will be assumed strictly quasi-concave throughout the paper.

Furthermore, we will assume that the functions $U$, $V$, $E$ and $T$ are twice continuously differentiable in their respective arguments and that the demand functions are continuously differentiable. This will allow us to discuss the comparative static properties of demand in terms of differentials.

Under such assumptions, the functions $U$, $V$, $E$ and $T$ are very useful to derive the empirical implications of consumer theory in terms of the demand functions $x(v,1)$ and $v(x,1)$. Such implications are well known and need not be discussed here [see, e.g., Phelps (1974), Deaton and Muellbauer (1980), Diewert (1974, 1982), Weymark (1980), Anderson (1980)]. However, these duality properties can be extended to cases where demands are function of both prices and quantities. One step in that direction is provided by the theory of household behavior under rationing. Although this theory is well

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1. $U(x)$ is strictly quasi-concave if $U[\alpha x_0 + (1-\alpha)x_1] > U(x_0)$ for $0 < \alpha < 1$ and $U(x_0) \geq U(x_1)$, or alternatively for a twice differentiable function, if $w'[\partial^2 U/\partial x^2] w \leq 0 \forall w, \partial U/\partial x w = 0$.

2. Note that assuming a continuously differentiable demand $x(v,1)$ derived from a strictly quasi-concave increasing differentiable utility function $U(x)$ is equivalent to the strong quasi-concavity of $U(x)$ [Barten and Bohm (1982)]. Thus, we assume throughout the paper that $U(x)$ is strongly quasi-concave, i.e., that $w'[\partial^2 U/\partial x^2] w < 0 \forall w \neq 0$, $\partial U/\partial x w = 0$. In this case, the associated indirect utility function $V(v)$ is strongly quasi-convex (see Corollary 1 in the appendix for a proof).

3. When prices are normalized ($\psi = p/y$), the homogeneity restriction is automatically satisfied. In that case, for the quantity dependent demand functions $x(v,1)$, the empirical implications of the theory reduce to the adding-up restrictions obtained by differentiating the budget constraint $\psi_x = 0$, $\psi(\partial x/\partial v) + x' = 0$, and the symmetry restriction where the Slutsky matrix $\partial x/\partial v = \partial x/\partial v - (\partial x/\partial v)nx'$ is symmetric and negative semi-definite. Similarly, for the price dependent demand functions $v(x,1)$, the theoretical restrictions are the adding-up condition $x'(\partial v/\partial x) + \psi' = 0$ and the symmetry restriction where the Antonelli matrix $\partial v/\partial x = \partial v/\partial x - (\partial v/\partial x)xv'$ is symmetric and negative semi-definite. Although these restrictions can be alternatively expressed in terms of nominal prices and income (i.e., without normalizing prices), we chose to use the normalized form in order to illustrate the symmetry between quantity and price dependent demand functions.
developed [Tobin and Houthakker (1950–1951), Pollak (1969), Howard (1977), Latham (1980), Neary and Roberts (1980)], it will be useful to review some of its implications for two reasons. First, mixed demand functions are closely related to conditional or rationed demand functions and we will take advantage of their relationships in the derivation of our results. Second, this will give us the opportunity to present a more unified treatment of demand theory.

3. Conditional demands

Conditional demand functions are appropriate tools for analyzing consumer behavior in a number of situations. Such situations include consumer rationing [Pollak (1969), Howard (1977), Latham (1980), Deaton (1981), Neary and Roberts (1980)], the distinction between short-run and long-run consumer behavior, the presence of non-market goods [e.g., Cornes (1980), Guernerie (1981)] and the analysis of the effects of leisure on consumption [Deaton and Muellbauer (1981)]. In order to discuss conditional demands, consider the partition of the commodity set into two subsets, and denote \( x = (x_1, x_2) \) and \( p = (p_1, p_2) \). We will assume that \( x_2 \) is a vector of goods imposed on the consumer and we will limit our attention to the feasible case where \( p_2 x_2 < y \). Then the utility maximizing consumer problem is

\[
\bar{V}(v_1, v_2, x_2, 1) = \max_{x_1} U(x_1, x_2) \quad \text{subject to} \quad v_1 x_1 + v_2 x_2 = 1,
\]

where \( v_1 = p_1/y \) and \( v_2 = p_2/y \). The solution of this optimization is rationed or conditional demand function \( \bar{x}_i(v_1, v_2, x_2, 1) \) which involves the normalized price vectors \( v_1 \) and \( v_2 \) as well as the quantities \( x_2 \) [Pollak (1969)]. Expression (7), which is the conditional equivalent of (1), also defines the conditional indirect utility function \( \tilde{V} \). As the indirect utility function \( V \), \( \bar{V}(v_1, v_2, x_2) \) is continuous, decreasing and strongly quasi-convex in \( v_1 \). It is also decreasing in \( v_2 \) and strictly concave in \( x_2 \).

\(^4\)Note that our definition of conditional indirect utility differs slightly from the definition given by Epstein (1975), Diewert (1978, 1982) or Blackorby et al. (1978, p. 182). Their definition is as follows: \( \bar{V}(v_1, x_2, \beta) = \max x_1, U(x_1, x_2) \), subject to \( v_1 x_1 = \beta \), where \( \beta = 1 - v_2 x_2 \) is the budget share of the commodity group \( x_1 \), and the corresponding conditional demand function is \( \bar{x}_1(v_1, x_2, \beta) \).

Such formulation has proved useful in the discussion of separability [Pollak (1969)] and aggregation, \( \bar{V} \) being called the variable indirect utility function by Epstein, and the aggregated utility function by Diewert (1978). The relationship between the two definitions are given by, e.g., Pollak \( \bar{x}_1(v_1, v_2, x_2, 1) = \bar{x}_1(v_1, x_2, 1 - v_2 x_2) \) and \( \bar{V}(v_1, v_2, x_2, 1) = \tilde{V}(v_1, x_2, 1 - v_2 x_2) \). Another notable difference between \( V \) and \( \bar{V} \) is that, while \( V \) is quasi-concave in \( x_2 \) [see Diewert (1978)], \( \bar{V} \) is a concave function of \( x_2 \) (see Proposition 2 in the appendix for a proof).

\(^5\)\( \tilde{V} \) is decreasing in \( v_2 \) since, from the envelope theorem, \( \partial \tilde{V}/\partial v_2 = \lambda x_2 \leq 0 \) given that the marginal utility of income, \( \lambda \), is positive. The strict concavity of \( \tilde{V} \) in \( x_2 \) follows from Proposition 2 in the appendix.
By analogy to (2), (3) and (5), we can similarly define
— the conditional direct utility function

$$U(v_1, x_1, x_2, 1) = \min_{v_2} V(v_1, v_2, 1) \text{ subject to } v_1 x_1 + v_2 x_2 = 1,$$

where the solution $v_2(x_1, x_2, 1)$ is the conditional (normalized) inverse demand function for $v_2$. It can be interpreted as the ordinary virtual price or shadow price as $\bar{v}_2$ is the price vector that would induce an unconstrained household to purchase the quantities $x_1$ and $x_2$ given the price vector $v_1$. By analogy with the properties of $\bar{V}$, $U(v_1, x_1, x_2)$ can be shown to be increasing and strongly quasi-concave in $x_2$, decreasing in $x_1$, and strictly convex in $v_1$.

— the conditional (normalized) expenditure function

$$E(v_1, v_2, x_2, U^0) = \min_{x_1} v_1 x_1 + v_2 x_2 \text{ subject to } U(x_1, x_2) = U^0,$$

where the solution $\bar{x}_1(x_2, v_1, U^0)$ is the conditional compensated demand function, which holds the consumer on a given utility level $U^0$. Note that $\bar{x}_1$ is not a function of $v_2$. As in (3), the conditional expenditure function $E$ is increasing in $U^0$, linear homogenous, increasing and concave in $v_1$ and $v_2$. It is also convex in $x_2$ [see (20) below].

— the conditional transformation function

$$T(v_1, x_1, x_2, V^0) = \min_{v_2} v_1 x_1 + v_2 x_2 \text{ subject to } V(v_1, v_2) = V^0,$$

where the solution $\bar{v}_2(x_2, v_1, V^0)$ is the conditional compensated inverse demand function for $v_2$, also called the compensated virtual or shadow price function by Neary and Roberts. Note that $\bar{v}_2$ is not a function of $x_1$. As in (5), and by analogy with the properties of $\bar{E}$, the conditional transformation function (10) can be shown to be decreasing in $V^0$, linear homogenous, increasing and concave in $x_1$ and $x_2$. It is also convex in $v_1$ [see (24) below].

Using the above conditional functions, a number of duality results follow.

$^6$ If $E$ is increasing in $v_1$ and $v_2$, since, from the envelope theorem, $\partial E/\partial v_1 = \bar{x}_1$, $\partial E/\partial v_2 = \bar{x}_2$. The concavity of $E$ in $(v_1, \bar{v}_2)$ then follows from the negative semi-definiteness of the matrix

$$\frac{\partial^2 E}{\partial v_1^2} = \begin{bmatrix} \bar{x}_1 & 0 \\ 0 & \partial^2 E \end{bmatrix}$$

(See Corollary 2 for the appendix for a proof.)

$^7$ Note that $\bar{v}_2$ could alternatively be obtained by minimizing (9) with respect to $x_2$, yielding the first order conditions $\bar{x}_2 = \lambda \cdot \partial U(x_1, x_2) / \partial x_2$, where $\lambda$ is the marginal normalized cost of utility in (9). Indeed, in both cases, $\bar{v}_2$ is the normalized price vector that corresponds to minimal expenditures and keeps the consumer on a given level of utility.
Indeed, from the envelop theorem, it can be shown that\(^8\)

\[
\begin{align*}
\bar{x}_1' &= \frac{\partial \bar{V}}{\partial v_1} v_1 + \frac{\partial \bar{V}}{\partial v_2} v_2, \\
\bar{v}_2' &= \frac{\partial \bar{U}}{\partial x_2} x_2,
\end{align*}
\]

(11), (12)

The above four expressions correspond respectively to Roy, Wold, Hotelling–Shepard and Shepard–Hanoch theorem [see, e.g., Weymark (1980)] in the context of conditional demand functions. Furthermore, the envelope theorem implies

\[
\frac{\partial \bar{E}}{\partial v_2} = x_2.
\]

(15)

and

\[
\frac{\partial \bar{E}}{\partial x_2} = (v_2 - \lambda \partial U(x_1, x_2)/\partial x_2)' = [v_2 - \bar{v}_2]'.
\]

(16)

This result, also obtained by Neary and Roberts (1980, p. 30, eq. (16)) follows from footnote 7. It can be similarly shown that

\[
\frac{\partial \bar{T}}{\partial x_1} = v_1' \quad \text{and} \quad \frac{\partial \bar{T}}{\partial v_1} = [x_1 - \bar{x}_1]'.
\]

(17), (18)

These results are useful to derive the comparative static implications of conditional demands. Using the ‘Instant Slutsky’ approach (Cook), consider the following identities: \(\bar{x}_1'[x_2, v_1, U^0] = \bar{x}_1[x_2, v_1, v_2, \bar{E}(x_1, v_1, v_2, U^0)]\), and \(\bar{v}_2'[x_2, v_1, V^0] = \bar{v}_2[x_1, x_2, v_1, T(x_1, x_2, v_1, V^0)]\), where the function \(E\) and \(T\) play the role of compensation functions, keeping the consumer on a given level of utility. By differentiation, and making use of (13) through (18), this gives

\(^8\)Writing \(\bar{V} = \bar{V}(v_1, v_2, x_2, k)\) where \(k = 1\), it is clear from (7) that \(\bar{V}\) is homogenous of degree zero in \(v_1, v_2\) and \(k\), i.e., \((\partial \bar{V}/\partial k)k + (\partial \bar{V}/\partial v_1)v_1 + (\partial \bar{V}/\partial v_2)v_2 = 0\), implying that \((\partial \bar{V}/\partial k) = -(\partial \bar{V}/\partial v_1)v_1 - (\partial \bar{V}/\partial v_2)v_2\). But, from the envelope theorem,

\[
\frac{\partial \bar{V}}{\partial v_1} = \frac{\partial \bar{V}}{\partial k} \bar{x}_1 = \left[\frac{\partial \bar{V}}{\partial v_1} v_1 + \frac{\partial \bar{V}}{\partial v_2} v_2\right] \bar{x}_1,
\]

thus proving (11). A similar procedure can be followed to prove (12).
\[
\frac{\partial^2 E}{\partial v_1^2} = \frac{\partial^2 \bar{E}}{\partial v_1} = \frac{\partial^2 \bar{E}}{\partial v_1} + \frac{\partial^2 \bar{E}}{\partial T^2 \bar{v}_2^2} \tag{19}
\]

which is the Slutsky matrix, symmetric by Young theorem and negative semi-definite by the concavity of \( \bar{E} \) in \( v \),

\[
\frac{\partial^2 \bar{E}}{\partial x_2^2} = -\frac{\partial \bar{v}_2}{\partial x_2} = - \left[ \frac{\partial \bar{v}_2}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial T \bar{v}_2} \right] \tag{20}
\]

which is symmetric, and positive semi-definite by (23) below, implying that \( \bar{E} \) is convex in \( x_2 \),

\[
\frac{\partial^2 \bar{E}}{\partial v_1 \partial x_2} = \frac{\partial \bar{v}_1}{\partial v_1} = \frac{\partial \bar{v}_1}{\partial v_1} + \frac{\partial \bar{v}_1}{\partial T \bar{v}_2} \tag{21}
\]

\[
\frac{\partial^2 \bar{E}}{\partial v_1 \partial v_2} = 0 = \frac{\partial \bar{v}_1}{\partial v_1} = \frac{\partial \bar{v}_1}{\partial v_1} + \frac{\partial \bar{v}_1}{\partial T \bar{v}_2} \tag{22}
\]

which are equations (26) and (27) in Neary and Roberts (1980, p. 34),

\[
\frac{\partial^2 \bar{T}}{\partial x_2^2} = \frac{\partial \bar{v}_2}{\partial x_2} = \frac{\partial \bar{v}_2}{\partial x_2} + \frac{\partial \bar{v}_2}{\partial T \bar{v}_2} \tag{23}
\]

which is the Antonelli matrix, symmetric and negative semi-definite from the concavity of \( \bar{T} \) in \( x \),

\[
\frac{\partial^2 \bar{T}}{\partial v_1^2} = -\frac{\partial \bar{v}_1}{\partial v_1} = - \left[ \frac{\partial \bar{v}_1}{\partial v_1} + \frac{\partial \bar{v}_1}{\partial T \bar{x}_1} \bar{x}_1' \right] \tag{24}
\]

which is symmetric, and positive semi-definite from (19), implying that \( \bar{T} \) is convex in \( v_1 \),

\[
\frac{\partial^2 \bar{T}}{\partial x_2 \partial v_1} = \frac{\partial \bar{v}_2}{\partial v_1} = \frac{\partial \bar{v}_2}{\partial v_1} + \frac{\partial \bar{v}_2}{\partial T \bar{x}_1} (x_1 - \bar{x}_1)' = - \left[ \frac{\partial \bar{v}_1}{\partial x_2} \bar{x}_1' \right] = - \left[ \frac{\partial \bar{v}_1}{\partial x_2} + \frac{\partial \bar{v}_1}{\partial T \bar{x}_1} (v_2 - \bar{v}_2)' \right] \tag{25}
\]

and finally

\[
\frac{\partial^2 \bar{T}}{\partial x_2 \partial x_1} = 0 = \frac{\partial \bar{v}_2}{\partial x_1} = \frac{\partial \bar{v}_2}{\partial x_1} + \frac{\partial \bar{v}_2}{\partial T \bar{v}_1} \tag{26}
\]
Except for the familiar homogeneity and adding $u$ restrictions, the above results characterize the empirical implications of consumer theory or conditional demands. By relating the slopes of the compensated demand functions $\tilde x_1$ and $\tilde x_2$ to the uncompensated slopes of $\check x_1$ and $\check x_2$, they extend a number of results obtained by Pollak or Neary and Roberts. In particular, eqs. (19)–(26) express the compensated slopes of the constant utility demand functions (or substitution effects) as the sum of uncompensated slopes (or total effects) and of a ‘scale effect’. This scale effect for the demand function $\check x_1$, $[(\partial \check x_1/\partial E)\check x_1]$, in (19), $[(\partial \check x_1/\partial T)\check x_1]$, in (21) and $[(\partial \check x_1/\partial E)\check x_1]$, in (22), corresponds to the well-known income effect and is equivalent to a rescaling of the price vector $v$. Similarly, the scale effect for the demand function $\check v_2$, $[(\partial \check v_2/\partial T)\check v_2]$, in (23), $[(\partial \check v_2/\partial T)(x_1 - x_2)]$, in (25) and $[(\partial \check v_2/\partial T)\check v_1]$, in (26), is equivalent to a rescaling of the quantity vector $x$. [Deaton (1979). Anderson (1980)]. Note that these properties, derived in the context of the conditional demands $\check x_1$ and $\check x_2$ include as special cases the properties of unconditional demands $x(u)$ and $v(x)$ when $x_1 = x$ or $v_2 = v$.

The above results can be very useful in empirical work. Given the nature of the objective functions (7) and (8), it can be shown that $\partial x_1/\partial E = -(\partial x_1/\partial v)v$ and $\partial x_2/\partial T = -(\partial x_2/\partial x)x$. By substituting these two expressions in (19)–(26), we obtain behavioral restrictions that are expressed in a form that can be tested and/or imposed in empirical models. Also, the relationships between the conditional demand function $\check x_1$ and the virtual price function $\check v_2$ can be of interest in cases where the latter is not directly estimable. Finally, the above results are of interest since they are related to the behavioral implications of Samuelson’s mixed demand functions, as we now proceed to show.

4. Mixed demands

The conditional demand $\check x_1$ just discussed are functions of both the prices and quantities of the ‘rationed’ commodities $x_2$. This implicitly assumes that the rationed commodities may not be consumed in an optimum way, i.e., the quantities $x_2$ may differ from the quantities that would be consumed in the absence of rationing. However, there is an alternative way of specifying demand functions involving both prices and quantities without assuming possible sub-optimality of consumption. There are the mixed demand functions proposed by Samuelson (1965). Samuelson defines such functions as the solution of the following primal-dual problem [Samuelson (1965, p. 791)]

$^9$While the homogeneity restriction is always satisfied when prices are normalized, the adding-up restrictions obtained by differentiating the budget constraint $r x = 1$ are

\[ v_1 \frac{\partial \check x_1}{\partial v} + x' = 0, \quad v_1 \frac{\partial \check x_1}{\partial x_2} + v_2' = 0, \quad x_1' \frac{\partial \check v_2}{\partial x} + v' = 0, \quad x_2' \frac{\partial \check v_2}{\partial x_2} + x_1 = 0 \]

$^{10}$This follows from the homogeneity of degree zero of $\check x_1$ in $E$ and $v$, and of $\check v_2$ in $T$ and $x$. 

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\[ 0 = \max_{x_1, x_2} U(x_1, x_2) - V(v_1, v_2) \quad \text{subject to} \quad v_1 x_1 + v_2 x_2 = 1, \tag{27} \]

where \( U \) is the direct utility function, \( V \) is the indirect utility function defined in section 2, and the subscripts 1 and 2 denote some partition of the commodity set \( x = (x_1, x_2) \) with corresponding prices \( v = p/y = (v_1, v_2). \)

The solutions of (27) are the quantity dependent mixed demand functions \( \tilde{x}_1(x_2, v_1, 1) \) and price dependent mixed demand functions \( \tilde{v}_2(x_2, v_1, 1). \) In contrast to the conditional demands discussed in section 3, they assume utility maximization with respect to all the commodities present in the budget constraint. As such, these mixed demands are always functions of a mixed set of \( n \) prices or quantities. They include, as special cases, the Marshallian demand functions \( x(v, 1) \) when \( x = x_1 \), as well as the inverse demand functions \( v(x, 1) \) when \( v = v_2 \) (see section 2).

Denoting by \( \lambda \) the Lagrange multiplier for the budget constraint in (27), the envelope theorem implies that, at the optimum [Samuelson (1965, p. 792)]

\[ 0 = -\partial U(x_1, x_2)/\partial x_2 - \lambda v_2, \tag{28} \]

\[ 0 = -\partial V(v_1, v_2, 1)/\partial v_1 - \lambda x_1. \tag{29} \]

Also, denoting \( \tilde{U}(x_2, v_1, 1) = U(\tilde{x}_1(x_2, v_1, 1), x_2) \) and \( \tilde{V}(x_2, v_1, 1) = V(\tilde{v}_2(x_2, v_1, 1), x_2) \), it is clear from (27) that \( \tilde{U}(x_2, v_1, 1) = \tilde{V}(x_2, v_1, 1). \) These results are now used to derive the theoretical implications of mixed demand specifications.

First, note that the optimization (27) can be decomposed into the two following sub-problems:

\[ \max_{x_1, x_2} U(x_1, x_2) + \max \left\{ -V(u) \text{ s.t. } x = 1 \right\} = \max \left\{ -U(x_1, x_2, 1) \right\}, \text{ or as} \]

\[ \max_{v_1, v_2} -V(u) + \max \left\{ U(x) \text{ s.t. } v = 1 \right\} = \max \left\{ U(x) - V(u) \right\}. \]

These relationships establish a clear link between conditional and mixed functions. Such relationships will be further explored in section 5.

\[ 11 \text{ Note that (27) can be alternatively expressed as} \]

\[ \max_{x_1, x_2} U(x_1, x_2) + \max \left\{ -V(u) \text{ s.t. } x = 1 \right\} = \max \left\{ -U(x_1, x_2, 1) \right\}, \text{ or as} \]

\[ \max_{v_1, v_2} -V(u) + \max \left\{ U(x) \text{ s.t. } v = 1 \right\} = \max \left\{ U(x) - V(u) \right\}. \]

These relationships establish a clear link between conditional and mixed functions. Such relationships will be further explored in section 5.

\[ 12 \text{ Note, although } U(x) \text{ is strongly quasi-concave and } V(u) \text{ is strongly quasi-convex, this does not imply} \]

\( \text{that the primal-dual function } \{U(x) - V(u)\} \text{ is strongly quasi-concave. Indeed, the Hessian of } \{U(x) - V(u)\} \text{ can only be shown to be negative semi-definite subject to constraint [Silverberg (1974)]}, \text{ implying that the maximum in (27) may not be unique. Indeed, as noted by } \]

\( \text{Samuelson (1965), because of the Giffen phenomenon, a particular demand function may not be a monotonic function of its own price, thus preventing the global invertibility of that function.} \)

\( \text{In that case, the analysis needs to be restricted to a domain where the demand functions are single valued, and the solution of the maximization problem (27) is unique. For the simplicity of exposition, we will limit our discussion to the domain of } x_1 \text{ and } v_2 \text{ where (27) has a unique solution. The existence of mixed demand systems is further discussed in footnote 13.} \)
\[ \bar{U}(x_2,v_1,1) = \max_{x_1} U'(x_1, x_2) \text{ subject to } v_1' x_1 + x_2' \bar{v}_2(x_2,v_1,1) = 1 \] (30)

and

\[ \bar{V}(x_2,v_1,1) = \min_{v_2} V(v_1,v_2) \text{ subject to } v_1 \bar{x}_1(x_2,v_1,1) + v_2' x_2 = 1 \] (31)

with \( \bar{x}_1 \) and \( \bar{v}_2 \) being obviously the solutions of (30) and (31) respectively. Expressions (30) and (31) define a mixed utility function, \( \bar{U} = \bar{V} \), which has an intermediate position between the direct and indirect utility functions \( U \) and \( V \). In a similar fashion, generalizing concepts discussed in section 2, we can define the mixed expenditure function \( \bar{E} \) and the mixed transformation function \( \bar{T} \) as follows:

\[ \bar{E}(x_2,v_1,U^0) = \min_{x_1} v_1' x_1 + x_2' \psi_3(x_2,v_1,\bar{E}) \text{ subject to } U(x_1,x_2) = U^0 \] (32)

and

\[ \bar{T}(x_2,v_1,U^0) = \min_{v_2} v_1 \bar{x}_1(x_2,v_1,\bar{T}) + v_2' x_2 \text{ subject to } V(v_1,v_2) = U^0, \] (33)

where the solutions are respectively the compensated mixed demands \( \bar{x}_1'(x_2,v_1,U^0) \) and \( \bar{v}_2'(x_2,v_1,U^0) \). Comparing (32) and (33) with expressions (9) and (10), it is clear that \( \bar{x}_1' = \bar{x}_1' \) and \( \bar{v}_2' = \bar{v}_2' \), i.e., that the compensated conditional demand functions are identical to the compensated mixed demands functions. This is a very useful result since we have already derived the theoretical implications of conditional demand (see section 3).

This indicates that the compensated mixed demand functions \( \bar{x}_1' \) and \( \bar{v}_2' \) always exist given our assumptions. As suggested by a reviewer, a proof of their existence can be obtained by inverting the compensated demand functions \( x'(v_1,v_2,U) \) or \( \psi'(x_1,x_2,U) \). To illustrate, consider the Slutsky matrix \( \partial x'/\partial v \) which is symmetric, negative semi-definite and of rank \( (n-1) \) (see Corollary 2 in appendix A). Defining the Slutsky submatrices \( h_{ij} = \partial x_{ij}/\partial v_{ij} \), it follows that \( h_{22} \) is a symmetric negative definite and invertible matrix. The function \( x'(v_1,v_2,U) \) can therefore always be inverted into the mixed functions \( \bar{x}_1'(v_1,x_2,U) \) and \( \bar{v}_2'(v_1,x_2,U) \). Using the property of an inverse function, it can be shown that [e.g., Samuelson (1960)]

\[
\begin{bmatrix}
\partial x_1 \\
\partial v_1 \\
\partial v_2 \\
\partial \bar{v}_1 \\
\partial \bar{v}_2 \\
\partial \bar{x}_2
\end{bmatrix} =
\begin{bmatrix}
h_{11} & h_{12} & h_{21} & h_{22} & h_{22}^{-1} \\
h_{12} & h_{11} & h_{21} & h_{22} & h_{22}^{-1} \\
h_{21} & h_{21} & h_{11} & h_{12} & h_{12}^{-1} \\
h_{22} & h_{22} & h_{12} & h_{11} & h_{11}^{-1} \\
h_{22} & h_{22} & h_{12} & h_{11} & h_{11}^{-1} \\
h_{22} & h_{22} & h_{12} & h_{11} & h_{11}^{-1}
\end{bmatrix}
\]

This gives the properties of the compensated mixed demand functions \( \bar{x}_1' \) and \( \bar{v}_2' \). The symmetry relationship \( \partial x_1'/\partial v_2 = -\partial v_2'/\partial v_1 \) holds. Since \( \bar{x}_1' = \bar{x}_1' \) and \( \bar{v}_2' = \bar{v}_2' \), these properties are of course the same we derived in the context of conditional functions [expressions (19) to (25)].
To make use of such results, we need to relate the compensated mixed demands \( \bar{x}_1 \) and \( \bar{v}_2 \) to the ordinary mixed demands \( \bar{x}_1 \) and \( \bar{v}_2 \). This necessitates the further investigation of the two compensation functions \( E \) and \( T \) which, by definition, must satisfy respectively \( U[x_2, v_1, E(x_2, v_1, U^0)] = U^0 \) and \( V[x_2, v_1, T(x_2, v_1, U^0)] = U^0 \). Differentiating these two expressions, and using (28), (29) and the envelope theorem, we obtain

\[
\begin{align*}
\frac{\partial U}{\partial x_2} &= 0 = \lambda \left[ \bar{v}_2 - x'_2 \frac{\partial \bar{v}_2}{\partial x_2} - \bar{v}_2 + \frac{\partial E}{\partial x_2} - x'_2 \frac{\partial \bar{v}_2}{\partial E} \frac{\partial E}{\partial x_2} \right], \quad (34a) \\
\frac{\partial U}{\partial v_1} &= 0 = \lambda \left[ -\bar{x}_1 - x'_2 \frac{\partial \bar{x}_1}{\partial v_1} - x'_2 \frac{\partial \bar{v}_2}{\partial E} \frac{\partial E}{\partial v_1} \right], \quad (34b) \\
\frac{\partial V}{\partial x_2} &= 0 = \lambda \left[ -\bar{v}_2 - v'_1 \frac{\partial \bar{v}_2}{\partial x_2} + \frac{\partial T}{\partial x_2} - v'_1 \frac{\partial \bar{v}_2}{\partial T} \frac{\partial T}{\partial x_2} \right], \quad (34c) \\
\frac{\partial V}{\partial v_1} &= 0 = \lambda \left[ -\bar{x}_1 - x'_2 - v'_1 \frac{\partial \bar{x}_1}{\partial v_1} + v'_1 \frac{\partial \bar{v}_2}{\partial T} \frac{\partial T}{\partial v_1} \right]. \quad (34d)
\end{align*}
\]

Expressions (34) define implicitly the characteristics of the two compensation functions, \( E \) and \( T \), which both keep the consumer on a given utility level \( U^0 \), but each in a different way. The mixed expenditure function \( E \) compensates basically by changing or 'rescaling' consumer income. In contrast, the mixed transformation function \( T \) compensates by rescaling the vectors \( v_1 \) and \( x_2 \).

Note that the functions \( E \) and \( T \) are not always defined. It follows from (34a) or (34b) that the mixed expenditure function \( E \) does not exist when \( x'_2 \partial \bar{v}_2 / \partial E = 1 \). Similarly, (34c) or (34d) implies that the mixed transformation function \( T \) does not exist when \( v'_1 \partial \bar{x}_1 / \partial T = 1 \). However, from the adding-up restriction, \( v'_1 \partial \bar{x}_1 / \partial T + x'_2 \partial \bar{v}_2 / \partial E = 1 \) when evaluated at \( T = E = 1 \), it is clear that when one of the two functions \( E \) or \( T \) does not exist, the other one is necessarily well defined. For example, in the pure quantity dependent case (\( x = x_1 \)), only the expenditure function exists. Similarly, in the pure price dependent case (\( x = x_2 \)), only the transformation function exists. Thus, in these two special cases well developed in the literature [e.g., Weymark (1980)], there is no possible ambiguity concerning the choice of the compensation function.

In the general mixed demand case where \( x = (x_1, x_2) \), both functions will exist if \( v'_1 \partial \bar{x}_1 / \partial T \neq 1 \) and \( x'_2 \partial \bar{v}_2 / \partial E \neq 1 \). Although these two conditions are not guaranteed by the theory, they are likely to be met in most empirical situations. Assuming that these two conditions are satisfied, expressions (34) become
\[
\frac{\partial E}{\partial x_2} = \left[ 1 - x_2 \frac{\partial \tilde{v}_2}{\partial E} \right]^{-1} x_2 \frac{\partial \tilde{v}_2}{\partial x_2},
\]
(35a)

\[
\frac{\partial E}{\partial v_1} = \left[ 1 - x_2 \frac{\partial \tilde{v}_2}{\partial E} \right]^{-1} \left[ \tilde{x}_1 + x_2 \frac{\partial \tilde{v}_2}{\partial v_1} \right],
\]
(35b)

\[
\frac{\partial \tilde{T}}{\partial x_2} = \left[ 1 - v'_1 \frac{\partial \tilde{x}_1}{\partial T} \right]^{-1} \left[ \tilde{v}_2 - v_1 \frac{\partial \tilde{x}_1}{\partial x_2} \right],
\]
(35c)

\[
\frac{\partial \tilde{T}}{\partial v_1} = \left[ 1 - v'_1 \frac{\partial \tilde{x}_1}{\partial T} \right]^{-1} v'_1 \frac{\partial \tilde{x}_1}{\partial v_1}.
\]
(35d)

Using these results in (35) and differentiating the identities \(\tilde{x}_1(x_2,v_1,U^0) = \tilde{x}_1(x_2,v_1,E(x_2,v_1,U^0))\) and \(\tilde{v}_2(x_2,v_1,U^0) = \tilde{v}_2(x_2,v_1,T(x_2,v_1,U^0))\), we obtain the following relationships between compensated and uncompensated mixed demand functions: \(^{14}\)

\[
\frac{\partial \tilde{x}_1}{\partial x_2} = \frac{\partial \tilde{x}_1}{\partial x_2} + \frac{\partial \tilde{x}_1}{\partial E} \left[ 1 - x_2 \frac{\partial \tilde{v}_2}{\partial E} \right]^{-1} \left[ \tilde{x}_1 + x_2 \frac{\partial \tilde{v}_2}{\partial v_1} \right],
\]
(36a)

\[
\frac{\partial \tilde{x}_1}{\partial v_1} = \frac{\partial \tilde{x}_1}{\partial v_1} + \frac{\partial \tilde{x}_1}{\partial E} \left[ 1 - x_2 \frac{\partial \tilde{v}_2}{\partial E} \right]^{-1} \left[ \tilde{x}_1 + x_2 \frac{\partial \tilde{v}_2}{\partial v_1} \right],
\]
(36b)

\[
\frac{\partial \tilde{v}_2}{\partial x_2} = \frac{\partial \tilde{v}_2}{\partial x_2} + \frac{\partial \tilde{v}_2}{\partial T} \left[ 1 - v'_1 \frac{\partial \tilde{x}_1}{\partial T} \right]^{-1} \left[ \tilde{v}_2 - v_1 \frac{\partial \tilde{x}_1}{\partial x_2} \right],
\]
(36c)

\(^{14}\)Note that the properties of the mixed demand functions \(\tilde{x}_1\) and \(\tilde{v}_2\) can also be derived directly from the relationships among the conditional demands \((\tilde{x}_1,\tilde{v}_2)\) and the mixed demand \((\tilde{x}_1,\tilde{v}_2)\). \(\text{[See (37) and (38) below.]}\) To illustrate, given \(k = 1\), consider the identity (37b): \(\tilde{x}_1(v_1,x_2,k) = \tilde{x}_1[v_1,\tilde{v}_2(v_1,x_2,k),x_2,k]\). Differentiating this relationship with respect to \(k\) and using (22), we have

\[
\frac{\partial \tilde{x}_1}{\partial v_2} = - \frac{\partial \tilde{x}_1}{\partial k} x_2' \quad \text{and} \quad \frac{\partial \tilde{x}_1}{\partial k} = \frac{\partial \tilde{x}_1}{\partial k} + \frac{\partial \tilde{x}_1}{\partial \tilde{v}_2} \frac{\partial \tilde{v}_2}{\partial k} = \frac{\partial \tilde{x}_1}{\partial k} \left(1 - x_2' \frac{\partial \tilde{x}_2}{\partial k} \right)
\]

thus implying that, for \(\partial \tilde{x}_2/\partial \tilde{v}_2 \neq 0\),

\[
\frac{\partial \tilde{x}_1}{\partial v_2} = - \left[1 - x_2' \frac{\partial \tilde{x}_2}{\partial k} \right]^{-1} \left[\frac{\partial \tilde{x}_1}{\partial k} \right] x_2'.
\]

Similarly, with \(\tilde{x}_1 = \tilde{x}_1\) and \(\tilde{v}_2 = \tilde{v}_2\), it follows from (21) and the differentiation of (37b) with respect to \(x_2\) that

\[
\frac{\partial \tilde{x}_1}{\partial x_2} = \frac{\partial \tilde{x}_1}{\partial x_2} + \frac{\partial \tilde{x}_1}{\partial \tilde{v}_2} \frac{\partial \tilde{v}_2}{\partial x_2}
\]

Combining the last two expressions yields

\[
\frac{\partial \tilde{x}_1}{\partial x_2} = \frac{\partial \tilde{x}_1}{\partial x_2} + \left[1 - x_2' \frac{\partial \tilde{x}_2}{\partial k} \right] \left[\frac{\partial \tilde{x}_1}{\partial x_2} \frac{\partial \tilde{v}_2}{\partial x_2} \right],
\]
which is eq. (36a). Eqs. (36b) through (36d) can be obtained in a similar fashion from the identities in (37) and (39) below.
Equations (36) express the compensated effects (or substitution effects) as the total (or uncompensated) effects (the first term on the right-hand side of (36) plus the 'scale effects' (the second term on the right-hand side of (36)).

Given that \( \tilde{x}_1 = \tilde{x}_1 \) and \( \tilde{y}_2 = \tilde{y}_2 \), and using the expressions (19) and (23) in section 3, it follows that \( \partial \tilde{x}_1 / \partial v_1 \) in (36b) and \( \partial \tilde{y}_2 / \partial x_2 \) in (36c) are symmetric and negative semi-definite matrices: they are the equivalent of the Slutsky matrix and the Antonelli matrix, respectively, in the context of mixed demand functions. Similarly, from (21) or (25), \( \partial \tilde{x}_1 / \partial x_2 = -\left[ \partial \tilde{y}_2 / \partial v_1 \right] \), implying that, except for the sign, compensated cross effects (36a) and (36d) are also symmetric in mixed demand specifications.

Again, except for the familiar homogeneity and adding-up restrictions,\(^{15}\) the above results characterize the behavioral implications of consumer theory when demand functions are specified in mixed form. These mixed demands provide added flexibility for empirical investigations of consumer behavior [Hein (1977)] since it does not constrain the model specification to be either in pure quantity dependent or price dependent form [as in Christensen and Manser (1977) or Anderson (1980)]. Furthermore, the empirical implications of the theory just derived can be tested and/or imposed in the modeling of consumer behavior using the mixed form approach.\(^{16}\) Thus, the added flexibility of mixed demands appear very attractive for the investigation of consumption decisions since it is obtained without sacrificing the elegance of the theory.

5. Relationships among alternative demand specifications

We have presented three alternative demand specifications: the 'pure' price dependent or quantity dependent demand \([x(v, 1) \text{ or } v(x, 1)]\), the conditional demand and the mixed demand. In this section, we investigate briefly some relationships that exist among these alternative specifications.

First, an important result obtained in section 4 is \( \tilde{x}_1 = \tilde{x}_1 \) and \( \tilde{y}_2 = \tilde{y}_2 \), i.e., the compensated mixed demands are identical to the compensated conditional demands. Given the relationships between slopes of compensated and uncompensated functions derived above, this implies that, in general,

\[ \frac{\partial \tilde{y}_2}{\partial v_1} = \frac{\partial \tilde{y}_2}{\partial v_1} + \frac{\partial \tilde{x}_1}{\partial T} \left[ 1 - v_1 \frac{\partial \tilde{x}_1}{\partial T} \right]^{-1} v_1 \frac{\partial \tilde{x}_1}{\partial v_1}. \] (36d)

\(^{15}\)For mixed demand functions, the adding-up restrictions are

\[ v' \frac{\partial \tilde{x}_1}{\partial v_1} + x_2 \frac{\partial \tilde{y}_2}{\partial v_1} + x_1 = 0 \quad \text{and} \quad v' \frac{\partial \tilde{x}_1}{\partial x_2} + x_2 \frac{\partial \tilde{y}_2}{\partial x_2} + v_2 = 0. \]

\(^{16}\)In order to make (36) fully operational for empirical research, it suffices to note from (27) that, because of the homogeneity of degree zero of \( \tilde{x}_1 \) in \( \tilde{E}_1 \), \( x_2 \) and \( v_1 \), and of \( \tilde{y}_2 \) in \( \tilde{T}_1 \), \( x_2 \), \( v_1 \), \( \partial \tilde{x}_1 / \partial \tilde{E} = -(\partial \tilde{x}_1 / \partial x_2) x_2 - (\partial \tilde{x}_1 / \partial v_1) v_1 \), and \( \partial \tilde{y}_2 / \partial \tilde{T} = -(\partial \tilde{y}_2 / \partial x_2) x_2 - (\partial \tilde{y}_2 / \partial v_1) v_1 \), and then substitute these expressions into (36).
uncompensated mixed demands \((\bar{x}_1, \bar{x}_2)\) differ from uncompensated conditional demands \((\bar{x}_1, \bar{x}_2)\) in a way which depends on how the corresponding scale effects differ.

Other relationships among the alternative specifications can be obtained by noting the following identities:

\[
\bar{x}_1[v_1, v_2, x_2(v_1, v_2, 1), 1] = x_1(v_1, v_2, 1),
\]

(37a)

\[
\bar{x}_1[v_1, \theta_2(v_1, x_2, 1), x_2, 1] = \bar{x}_1(v_1, x_2, 1),
\]

(37b)

\[
\bar{x}_1[x_2(v_1, v_2, 1), v_1, 1] = x_1(v_1, v_2, 1)
\]

(37c)

and

\[
\theta_2[x_1, x_2, v_1(x_1, x_2, 1), 1] = v_2(x_1, x_2, 1),
\]

(38a)

\[
\bar{\theta}_2[\bar{x}_1(x_2, v_1, 1), x_2, v_1, 1] = \bar{\theta}_2(x_2, v_1, 1),
\]

(38b)

\[
\bar{\theta}_2[x_2, v_1(x_1, x_2, 1), 1] = v_2(x_1, x_2, 1).
\]

(38c)

Differentiating \((37)\) and \((38)\) with respect to the arguments of the functions provides a simple and convenient way of obtaining relationships among pure, conditional and mixed demands. To illustrate the approach, differentiate \((37a)\) to obtain

\[
\frac{\partial x_1}{\partial v_1} = \frac{\partial \bar{x}_1}{\partial v_2} + \frac{\partial \bar{x}_1}{\partial x_2} \frac{\partial x_2}{\partial v_1},
\]

(39a)

\[
\frac{\partial x_1}{\partial v_2} = \frac{\partial \bar{x}_1}{\partial v_2} + \frac{\partial \bar{x}_1}{\partial x_2} \frac{\partial x_2}{\partial v_2}.
\]

(39b)

Such results may be useful in empirical work. Also, they can be used to further investigate the differences between pure and conditional demand. For example, the Le Chatelier principle [Pollak (1969), Neary and Roberts (1980)] can be easily obtained from \((39)\). Indeed, using the properties of the Slutsky matrix along with \((22)\) and \((39)\), it follows that

\[
\left[ \frac{\partial x_1^*}{\partial v_1} \right] = \frac{\partial x_1^*}{\partial v_2} = \frac{\partial \bar{x}_1^*}{\partial v_2} + \frac{\partial \bar{x}_1}{\partial x_2} \frac{\partial x_2^*}{\partial v_2} = \frac{\partial \bar{x}_1}{\partial x_2} \frac{\partial x_2^*}{\partial v_2}
\]

(40)

and

\[
\frac{\partial x_1^*}{\partial v_1} = \frac{\partial \bar{x}_1^*}{\partial v_1} + \frac{\partial \bar{x}_1}{\partial x_2} \frac{\partial x_2^*}{\partial v_1}.
\]

(41)
Substituting (40) into (41) yields

$$\frac{\partial x'_1}{\partial v_1} = \frac{\partial x'_1}{\partial v_1} + \frac{\partial x'_1}{\partial x_2} \frac{\partial x'_2}{\partial v_2} \left( \frac{\partial x'_1}{\partial x_2} \right)'$$

(42)

which is the Le Chatelier result: since the matrix \((\partial x'_1/\partial x_2)(\partial x'_2/\partial v_2)(\partial x'_1/\partial x_2)'\) is negative semi-definite, the difference between the matrices of rationed and unrationed compensated price derivatives \((\partial x'_1/\partial v_1 - \partial x'_1/\partial v_1)\) is positive semi-definite. Note that this simple proof of the Le Chatelier principle differs from the one found in Pollak or Neary and Roberts since it does not require that the matrix \(\partial x'_2/\partial v_2\) be non-singular [Pollak (1969, p. 76), Neary and Roberts (1980, p. 34)].

A similar Le Chatelier result holds on the price dependent side. Following the same procedure, it can be shown that the implicit price functions \(v_2\) and \(\tilde{v}_2\) satisfy

$$\frac{\partial v'_2}{\partial x_2} = \frac{\partial v'_2}{\partial x_2} + \frac{\partial v'_2}{\partial v_1} \frac{\partial v'_1}{\partial x_1} \left( \frac{\partial v'_2}{\partial v_1} \right)'$$

(43)

which implies that the difference between the compensated matrices \(\partial v'_2/\partial x_2\) and \(\partial v'_2/\partial x_2\) is positive semi-definite.

These results illustrate the relationships between pure and conditional demands. By differentiating (37) and (38), additional relationships among alternative specifications can be derived as well. Such derivations are straightforward and need not be presented here.

6. Summary

In this paper, the theory of household behavior has been extended by presenting its implications for alternative demand specifications. Besides the well known pure quantity dependent and price dependent cases \([x(v, 1)\) and \(v(x, 1)\), the theoretical implications of conditional demands \([\tilde{x}(v_1, v_2, x_2, 1)\) and \(\tilde{v}_2(v_1, x_1, x_2, 1)\)] have been generalized, and the theory of mixed demands \([\tilde{x}_1(x_2, v_1, 1)\) and \(\tilde{v}_2(x_2, v_1, 1)\)] has been developed. For each demand specification, the effects of a price or quantity change have been decomposed into a substitution effect (which is always symmetric and negative semi-definite) and a 'scale' effect. The close relationships existing among these alternative specifications provide an unified treatment of demand theory. Also, by deriving the empirical implications of the theory in each case, this paper provides added flexibility in the empirical analysis of consumer behavior.
Appendix A

Consider the maximization of the function \( f(x, \alpha) \) with respect to the \( n \)-vector \( x \) of decision variables, subject to the constraint \( g(x, \alpha) = 0 \), where \( \alpha \) is a \( m \)-vector of parameters, \( f \) and \( g \) are twice continuously differentiable functions, and \( g_x = \frac{\partial g}{\partial x} \) has rank 1. Assume the Lagrangean of this problem

\[
L = f(x, \alpha) + \lambda g(x, \alpha)
\]  

has a strict local interior maximum \( x(\alpha) \) corresponding to

\[
L_x = f_x + \lambda g_x = 0, \quad (A.2a)
\]

\[
L_\lambda = g = 0, \quad (A.2b)
\]

\[
u' L_{xx} \nu < 0 \quad \forall \nu \neq 0, \quad g_x \nu = 0, \quad (A.2c)
\]

where subscripts indicate derivatives, i.e., \( L_x = \frac{\partial L}{\partial x} \), \( L_{xx} = \frac{\partial^2 L}{\partial x^2} \), etc.

From the strong quasi-concavity assumption (A.2c), the bordered hessian

\[
H = \begin{bmatrix}
L_{xx} & g_x \\
g_x & 0
\end{bmatrix}
\]

is non-singular (Debreu). The classical comparative static results [Samuelson (1947, pp. 357-359)] are then given by

\[
\begin{bmatrix}
X_x \\
\lambda_x
\end{bmatrix} = - H^{-1} \begin{bmatrix}
L_{xx} \\
g_x
\end{bmatrix} = - \begin{bmatrix}
A \\
B^T
\end{bmatrix} \begin{bmatrix}
L_{xx} \\
g_x
\end{bmatrix}, \quad \text{where}
\]

\[
\begin{bmatrix}
A & B & C
\end{bmatrix} = - H^{-1}, \quad A, \ B \text{ and } C
\]

are \( n \times n, \ n \times 1 \) and \( 1 \times 1 \) matrices respectively, and \( X_x \) and \( \lambda_x \) are matrices of comparative static slopes. Now, consider the indirect objective function \( L^* = f[x(\alpha), \alpha] \). Using the envelope theorem (Silberberg), differentiating \( L^* \) with respect to \( \alpha \) at the optimum gives

\[
L_x^* = L_{x\alpha}, \quad (A.5a)
\]

\[
L_{xx}^* = L_{xx} + L_{x\alpha} X_x + L_{\alpha \alpha} \lambda_x. \quad (A.5b)
\]

The properties of the indirect objective function \( L^* \) are given in the following proposition:
Proposition 1. The maximization problem (A.1) implies that

\[ w' \left[ L_{xa}^a - L_{za}^a \right] w = w' \left[ L_{xa} g_x^a \right] \begin{bmatrix} X_a \\ \lambda_a \end{bmatrix} w \begin{cases} = 0 & \text{iff } \begin{bmatrix} L_{xa} & g_x^a \\ g_x & 0 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0, \\ > 0 & \text{otherwise,} \end{cases} \]

\[ w \neq 0, \quad g_a w = 0. \] (A.6)

Proof. Given \( g_a u_1 = 0 \), the strong quasi-concavity assumption (A.2c) can be written as

\[ -(u_1' u_2) H \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{cases} = 0 & \text{iff } u_1 = 0, \\ > 0 & \text{otherwise.} \end{cases} \] (A.7)

the relationship \( g_a = -g_x X_a \) obtained by differentiating the constraint \( g \) at the optimum, and defining \( (u_1' u_2) = w' (X_a' \lambda_a) \), it follows from (A.7) that there exists a vector \( w \neq 0 \) satisfying \( g_a w = 0 \), and

\[ w' \left[ X_a' \lambda_a \right] H \begin{bmatrix} X_a \\ \lambda_a \end{bmatrix} w \begin{cases} = 0 & \text{iff } X_a w = 0, \\ > 0 & \text{otherwise.} \end{cases} \] (A.8)

Also, from (A.4), we have

\[ -(X_a' \lambda_a) H \begin{bmatrix} X_a \\ \lambda_a \end{bmatrix} = -[L_{za} g_a] H^{-1} \begin{bmatrix} L_{za} \\ g_a \end{bmatrix} = \begin{bmatrix} g_x' \\ \lambda_a \end{bmatrix}. \]

Substituting this expression into (A.8) and using (A.5b) yields, given \( w \neq 0 \), \( g_a w = 0 \),

\[ w' \left[ L_{za}^a - L_{za}^a \right] w = w' \left[ L_{za} g_a^a \right] \begin{bmatrix} X_a \\ \lambda_a \end{bmatrix} w \begin{cases} = 0 & \text{iff } X_a w = 0, \\ > 0 & \text{otherwise.} \end{cases} \] (A.9)

In order to prove Proposition 1, it remains to be shown that \( X_a w = 0 \) iff

\[ \begin{bmatrix} L_{za} & g_x \\ g_x & 0 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = 0. \]

In order to do that, note from (A.4) that \( X_a \) can be expressed

\[ X_a = -AL_{xa} - Bg_a. \] (A.10)

Using the relationship \( g_a = -g_x X_a \) obtained by differentiating the constraint
At the optimum, (A.10) can be alternatively written as

\[
\begin{bmatrix}
-I + B & g_x \\
-B & g_x
\end{bmatrix}X_a = \begin{bmatrix} A & L_{xx} \\
B & g_x
\end{bmatrix}.
\]

It follows that

\[
\begin{bmatrix}
-I + B & g_x \\
-B & g_x
\end{bmatrix}X_a w = 0 \iff \begin{bmatrix} A & L_{xx} \\
B & g_x
\end{bmatrix}w = 0.
\]

(A.11)

Since it is clear that the $2n \times n$ matrix

\[
\begin{bmatrix}
-I + B & g_x \\
-B & g_x
\end{bmatrix}
\]

has rank $n$, its null space is empty. Thus, (A.11) can be alternatively expressed as

\[
X_a w = 0 \iff \begin{bmatrix} A & L_{xx} \\
B & g_x
\end{bmatrix}w = 0.
\]

(A.12)

It can be shown that $A$ and $B$ have rank $n-1$ and 1 [see Pauwels (1979)], implying that these matrices have a null space of dimensions 1 and 0 respectively. Also, from the inverse of the Hessian $H$ [see (A.3) and (A.4)], we have $Ag'_x = 0$, implying that the $(r \times 1)$ matrix $g'_x$ of rank 1 spans the null space of $A$. Since $B$ has an empty null space, then $Bg_x w = 0$ is equivalent to $g_x w = 0$. Similarly, from $AL_{xx} w = 0$, $L_{xx} w$ is a subspace of the null space of $A$. Since this null space is spanned by $g'_x$, then $AL_{xx} w = 0$ is equivalent to $L_{xx} w = -g'_x z$. Using these results, (A.12) becomes

\[
X_a w = 0 \iff \begin{bmatrix} L_{xx} & g'_x \\
g_x & 0
\end{bmatrix} \begin{bmatrix} w \\
z
\end{bmatrix} = 0.
\]

(A.13)

Combining (A.13) with (A.9) gives the desired result. Q.E.D.

Note that the results in Proposition 1 are stronger than those obtained by Silberberg or Hatta since they distinguish between the quasi-convexity and the strong quasi-convexity of the primal-dual function ($L^*$ $L$). More specialized results are presented in the following corollaries.

**Corollary 1.** If the problem is to maximize $f(x)$ subject to $k = x'x$, then (1) becomes

\[
L = f(x) + \lambda[k - a'x],
\]
where $\alpha > 0$ and $k$ is a non-zero constant. In this case,

$$w' L^*_w w = w' \left[ \begin{array}{cc} -\lambda & -x \\ x' & 0 \end{array} \right] \begin{array}{c} Xx \\ \lambda \end{array} \right] w > 0, \quad w \neq 0, \quad x' w = 0$$

implying that the indirect objective function $L^*$ is a strongly quasi-convex function of $\alpha$.

**Proof.** This follows directly from Proposition 1 by noting that $L_{xx} = -\lambda I$, $L_{xz} = 0$, $g_x = -x'$, $g_z = -\alpha'$, and that the matrix

$$\begin{bmatrix} -\lambda & -\alpha \\ -x' & 0 \end{bmatrix}$$

has an empty null space. Q.E.D.

**Corollary 2.** If the problem is to minimize $\alpha' x$ subject to $k = f(x)$, then (1) becomes

$$L = \alpha' x + \lambda [k - f(x)],$$

where $\alpha > 0$ and $k$ is some non-zero constant. In this case

$$w' L^*_w w = w' X_w w \leq 0, \quad w \neq 0$$

implying that indirect objective function $L^*$ is a concave function of $\alpha$, the matrix $X_w$ being symmetric, negative semi-definite and of rank $(n - 1)$.

**Proof.** This follows directly from Proposition 1 by changing the inequality sign (since we have a minimum instead of a maximum) and noting that $L_{xx} = 0$, $L_{xz} = I$, $g_x = 0$. Moreover, since the matrix

$$\begin{bmatrix} 1 & -f'_x \\ 0 & 0 \end{bmatrix}$$

has dimensions $(n + 1) \times (n + 1)$ and rank $n$, its null space is of dimension 1, implying $L^*$ is a concave function of $\alpha$, and $X_w$ has rank $(n - 1)$. Q.E.D.

**Appendix B**

Consider the maximization problem discussed in appendix A, with the Lagrangean

$$J := f(x, \alpha) + \lambda g(x, \alpha). \quad (B.1)$$
In addition, assume that $L$ is strongly quasi-concave in both $x$ and $x$, although only the vector $x$ is a vector of decision variables, the optimal decision being denoted by $x(\alpha)$. Thus, in addition to (A.2), we assume

$$(u'_1 u'_2) \begin{bmatrix} L_{xx} & L_{ax} \\ L_{ax} & L_{aa} \end{bmatrix} (u_1 u_2) < 0 \quad \forall (u'_1 u'_2) \neq 0, \quad (u'_1 u'_2) \begin{bmatrix} g'_a \\ g'_x \end{bmatrix} = 0. \quad (B.2)$$

This implies that the matrices

$$H = \begin{bmatrix} L_{xx} & g'x \\ g_x & 0 \end{bmatrix} \quad (B.3a)$$

and

$$M = \begin{bmatrix} L_{xx} & L_{ax} & g'_a \\ L_{ax} & L_{ax} & g'_x \\ g_a & g_x & 0 \end{bmatrix} - \begin{bmatrix} L_{ax} & L_{ax} & g'_a \\ L_{ax} & L_{xx} & g_x \\ g_x & H \end{bmatrix} \quad (B.3b)$$

are both non-singular [Debreu (1952)]. Denote the inverse of the matrix $M$ by

$$M^{-1} = \begin{bmatrix} D & E \\ E' & F \end{bmatrix}, \quad (B.4)$$

where $D$, $E$ and $F$ are $(m \times m)$, $m \times (n+1)$ and $(n+1) \times (n+1)$ matrices respectively. Given that $H$ is non-singular, it follows from the partitioned inverse of a matrix that

$$D^{-1} = L_{xx} - (L_{ax} g'_x) H^{-1} \begin{bmatrix} L_{xx} \\ g_x \end{bmatrix}. \quad (B.5)$$

The main result is then stated in the following proposition.

**Proposition 2.** Assuming (B.2), the maximization problem (B.1) implies that

$$_w^r L^* \in R < 0 \quad \forall w \neq 0,$$

i.e., the indirect objective function $L^* = f[x(x), \alpha]$ is a strictly concave function of $\alpha$.

**Proof.** From (A.5b), we have

$$L_{xx}^* = L_{xx}^* + \begin{bmatrix} L_{xx} & g'_x \end{bmatrix} \begin{bmatrix} x_x \\ \lambda_x \end{bmatrix}.$$
Using (A.4) and (B.5), this expression becomes

\[ L_{ax}^* = L_{ax} - \left[ L_{ax} g_a' \right] H^{-1} \begin{bmatrix} \ell_{xa} \\ g_a \end{bmatrix} = D^{-1}. \]  \hspace{1cm} (B.6)

To prove Proposition 2, it remains to be shown that \( D^{-1} \) (or \( D \)) is a negative definite matrix. In order to do that, rewrite (B.2) as

\[ (u_1 u_2 u_3) M \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} < 0 \quad \forall (u_1 u_2 u_3) \neq 0, \quad (u_1 u_2) \begin{pmatrix} g_a \\ g'_x \end{pmatrix} = 0. \]

Defining \((u_1 u_2 u_3) = (w_1 w'_1) M^{-1}\) where \(w_1\) and \(w'_2\) are \((m \times 1)\) and \((n+1) \times 1\) vectors respectively, and using (B.4), it follows that

\[ (w'_1 w'_2) \begin{bmatrix} D & E \\ E' & F \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} < 0 \quad \forall (w'_1 w'_2) \begin{bmatrix} D & E \\ E' & F \end{bmatrix} \begin{pmatrix} g_a \\ g'_x \end{pmatrix} = 0. \]  \hspace{1cm} (B.7)

Choosing \(w'_2 = 0\), (B.7) becomes

\[ w'_1 D w_1 < 0 \quad \forall w_1 \neq 0, \quad w'_1 \left| D g'_a + E \begin{pmatrix} g'_x \\ 0 \end{pmatrix} \right| = 0. \]  \hspace{1cm} (B.8)

But, from the definition of an inverse in (B.4), we have

\[ D g'_a + E \begin{pmatrix} g'_x \\ 0 \end{pmatrix} = 0 \]

implying from (B.8) that

\[ w'_1 D w_1 < 0 \quad \forall w_1 \neq 0, \]

i.e., that \( D \) (and therefore \( D^{-1} \)) is a negative definite matrix. This result along with (B.6) concludes the proof. \hspace{1cm} Q.E.D.

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