Appendix I: Using Summation Notation With GAMS

AI.1 Summation Mechanics................................................................. 1
  AI.1.1 Sum of an Item. .................................................................. 1
  AI.1.2 Multiple Sums ................................................................. 1
  AI.1.3 Sum of Two Items .......................................................... 2
AI.2 Summation Notation Rules............................................................ 2
  AI.2.1 For a Scalar Equation .................................................... 2
  AI.2.2. For a Family of Equations ............................................ 3
AI.4 Defining and Using Variables ..................................................... 6
AI.5 Equations ............................................................................. 7
AI.6 Cautions and Extensions .......................................................... 8
Appendix I: Using Summation Notation With GAMS

Summation notation is difficult for some students to use and follow. Here we present notes on the mechanics of summation notation usage and some rules for proper use. This discussion is cast within the GAMS framework with presentation equivalents of common summation expressions and error messages caused by improper summation. All of the GAMS statements used herein are shown in Table 1 and are in file NOTATION.

Al.1 Summation Mechanics

Summation notation is a short hand way of expressing sums of algebraic terms involving subscripted items. In order to cover the mechanics of summation notation it is useful to have a set of subscripted items and associated numerical values. Thus, let us define some data

\[
\begin{align*}
  x_1 &= 1 \\
  y_{11} &= 2 \\
  y_{12} &= 3 \\
  x_2 &= 2 \\
  y_{21} &= 4 \\
  y_{22} &= 1 \\
  x_3 &= 3 \\
  y_{31} &= 1 \\
  y_{32} &= 4. 
\end{align*}
\]

Now let us define a variety of summation expressions.

Al.1.1 Sum of an Item.

Suppose we wished to sum all values of \( x \). This would be written as

\[
\sum_{i=1}^3 x_i = x_1 + x_2 + x_3 = 1 + 2 + 3 = 6
\]

or in GAMS

\[
\text{SUM1} = \text{SUM}(I, X(I));
\]

For short hand purposes if \( i \) was to be summed over all possible values, we would write this as

\[
\sum_i x_i.
\]

We might also express a sum as follows which indicates all of the \( i \) are summed over except \( i=3 \)

\[
\sum_{i=1, i \neq 3} x_i = x_1 + x_2 = 1 + 2 = 3.
\]

In GAMS, this is more difficult to express where one has to write a conditional (\$) operation or define a subset as follows

\[
\text{SUM1} = \text{SUM}(I$(\text{ORD}(I\text{.NE.3})), X(I));
\]

or

\[
\text{SET SUBSETI(I) /1, 2/;}
\]

\[
\text{SUM1} = \text{SUM(SUBSETI, X(SUBSETI(I))});
\]

Al.1.2 Multiple Sums

Sums over two indices consider all combinations of those items
\[ \sum_{i} \sum_{j} y_{ij} = y_{11} + y_{12} + y_{21} + y_{22} + y_{31} + y_{32} = 15. \]

The equivalent GAMS expression is

\[
\text{SUM2} = \text{SUM}((I,J), Y(I,J));
\]

AI.1.3 Sum of Two Items

Suppose we wished to sum over two items completely where they shared a subscript

\[
\sum_{i=1}^{3} \left( x_{i} + \sum_{j=1}^{2} y_{ij} \right) = \sum_{i} x_{i} + \sum_{i} \sum_{j} y_{ij} = x_{1} + y_{11} + y_{12} + x_{2} + y_{21} + y_{22} + x_{3} + y_{31} + y_{32} = 21.
\]

The equivalent GAMS expression is as follows

\[
\text{SUM3} = \text{SUM}(I, X(I)+\text{SUM}(J, Y(I, J)));
\]

or

\[
\text{SUM3} = \text{SUM}(I, X(I)) + \text{SUM}((I,J), Y(I,J));
\]

On the other hand, if we wished to sum the results only for the \(i^{th}\) element and call it \(A_{i}\) then

\[
A_{i} = x_{i} + \sum_{j} y_{ij} = x_{i} + y_{i1} + y_{i2}
\]

or in GAMS

\[
A(I) = X(I) + \text{SUM}(J, Y(I,J));
\]

which would yield a vector \([6, 7, 8]\) of results.

Sums over common subscripts can be collapsed or taken apart

or

\[
\sum_{i} (x_{i} + z_{i}) = \sum_{i} x_{i} + \sum_{i} z_{i}
\]

\[
\text{SUM4} = \text{SUM}(I, X(I) + Z(I));
\]

or

\[
\text{SUM4} = \text{SUM}(I, X(I)) + \text{SUM}(I, Z(I));
\]

### AI.2 Summation Notation Rules

Certain rules apply when writing summation notation equations. The applicable rules depend on whether the final result is an unsubscripted scalar or a subscripted family of results determined by multiple equations.

#### AI.2.1 For a Scalar Equation

All subscripts must be dealt with in each term. Thus, it is proper to define the equation
\[ B_1 = \sum_{i} \sum_{j} \sum_{k} p_{ik} + \sum_{m} \sum_{n} q_{mn}. \]

However, the following equations are wrong
\[ B_2 = p_{ik} + q_{mn} \]
\[ B_3 = \sum_{j} \sum_{i} p_{ik} + \sum_{m} \sum_{n} q_{mn}. \]

In the case of the first equation, the result would really have the subscripts \(i,j,k,m,n\),
while the second equation result would have to have a \(k\) subscript on \(B_3\) or a sum over \(k\) to be proper.

Equivalent GAMS commands for the above equation expressions are

\[ \text{EQB1.. } B1 =E= \text{SUM}((I,J,K), P(I,J,K)) + \text{SUM}((M,N), Q(M,N)); \]
\[ \text{EQB2.. } B2 =E= P(I,J,K) + Q(M,N); \]
\[ \text{EQB3.. } B3 =E= \text{SUM}((I,J), P(I,J,K)) + \text{SUM}((M,N), Q(M,N)); \]

Here, the first equation expression is correct, while the last two equation expressions are incorrect. If you run GAMS with the above commands, you would encounter GAMS error messages $149 which says "UNCONTROLLED SET ENTERED AS CONSTANT" meaning that you have not somehow dealt with all the subscripts in the equation.

AI.2.2. For a Family of Equations

Several rules apply when one is working with a family of equations.

1. The members of the family must be specified with an indication of the subscripts which define each equation. This is done by indicating all the conditions for which the equations exist in a "for" condition. For example, suppose we define an equation which sets all \(C\)'s equal to 2. This is done by saying
   \[ C_i = 2 \quad \text{for all } i \quad \text{or} \quad C_i = 2 \quad \text{for } i = 1,2, \ldots n. \]

   Similarly, if we wish to set a 2 dimensional variable equal to a constant, we would state
   \[ D_{ij} = 2 \quad \text{for all } i \text{ and } j, \]

   while stating that for each row of the matrix \(E_{ij}\) we have the same values \(F_i\) is defined by
   \[ E1_{ij} = F_i \text{for all } i \text{ and } j. \]

   The equivalent GAMS commands for the above expressions are

   EQUATIONS
   EQC(I) \quad \text{EQUATION C}
   EQD(I,J) \quad \text{EQUATION D}
   EQE1(I,J) \quad \text{EQUATION E1;}
   EQC(I) \quad \text{C(I) =E= 2;}

   AI-3

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EQD(I,J).. D(I,J) =E= 2;
EQE1(I,J).. E1(I,J) =E= F(I);

On the other hand, it is wrong to state

\[ E_{2ij} = 2 \]

without conditions on i and j. The equivalent GAMS commands for the above incorrect expressions are

\[ \text{EQUATION} \]
\[ \text{EQE2} \]
\[ \text{EQUATION E2;} \]
\[ \text{EQE2..} \]
\[ \text{E2(I,J) =E= 2;} \]

Here you would get error message $149$ which says "UNCONTROLLED SET ENTERED AS CONSTANT."

2. When writing an equation with a for statement, all subscripts which are not in the for statement must be summed over. Consequently, it is proper to write

\[ \sum_j \sum_k p_{ijk} = G_{1i} \quad \text{for all } i \]
\[ \sum_k p_{ijk} = H_{1i} \quad \text{for all } i \text{ and } j \]

but improper to write

\[ p_{ijk} = G_{2i} \quad \text{for all } i \]
\[ \sum_k p_{ijk} = H_{2i} \quad \text{for all } i. \]

The equivalent GAMS commands for the above equations are

\[ \text{EQUATIONS} \]
\[ \text{EQG1(I)} \]
\[ \text{EQUATION G1} \]
\[ \text{EQH1(I,J)} \]
\[ \text{EQUATION H1} \]
\[ \text{EQG2(I)} \]
\[ \text{EQUATION G2} \]
\[ \text{EQH2(I)} \]
\[ \text{EQUATION H2;} \]
\[ \text{EQG1(I)..} \]
\[ G1(I) =E= \text{SUM((J,K), P(I,J,K))}; \]
\[ \text{EQH1(I,J)..} \]
\[ H1(I,J) =E= \text{SUM(K, P(I,J,K))}; \]
\[ \text{EQG2(I)..} \]
\[ G2(I) =E= \text{P(I,J,K)}; \]
\[ \text{EQH2(I)..} \]
\[ H2(I) =E= \text{SUM(K, P(I,J,K))}; \]

AI-4

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in which the first two equations are correct, while the last two equations are wrong and error messages $149$ "UNCONTROLLED SET ENTERED AS CONSTANT" would again be realized.

3. In any term of an equation, the result after executing the mathematical operations in that term must be of a dimension less than or equal to the family definition in the for statement.

For example, it is proper to write

$$\sum_j \sum_k p_{ijk} = L1 \quad \text{for all } i$$

$$\sum_j \sum_k r_{jkm} + \sum_j s_{jim} = N_{im} \quad \text{for all } i \text{ and } m$$

but wrong to write

$$p_{ijk} = L2 \quad \text{for all } i.$$

Thus, for the following expressions, the first two equations are appropriate but the last equation would give you error message $149$ "UNCONTROLLED SET ENTERED AS CONSTANT."

EQUATION

EQL1(I)...

LI(I) =E= SUM((J,K), P(IJK));

EQN(I,M)...

N(I,M) =E= SUM((J,K), R(I,J,K,M))

+ SUM(J, S(I,J,M));

EQL2(I)...

L2 =E= P(I,J,K);

4. When the dimension is less than the family definition this implies the same term appears in multiple equations. For example, in the equation

$$2 + \sum_j \sum_k p_{ijk} + \sum_j s_{jim} = O_{im} \quad \text{for all } i \text{ and } m,$$

the 2 term appears in every equation and the sum involving p is common when m varies.

Equivalent GAMS commands are as follows

EQUATION

EQO(I,M) EQUATION O;

EQO(I,M). 2 + SUM((J,K), P(I,J,K)) + SUM(J, S(I,J,M)) =E= O(I,M);
5. In an equation you can never sum over the parameter that determines the family of equations. It is certainly wrong to write
\[ \sum_i \sum_j \sum_k p_{ijk} = W_i \text{ for all } i. \]

Or, equivalently, the following expressions are wrong and will result in error message $125$
which says "SET IS UNDER CONTROL ALREADY."
EQW(I)... W(I) = E = SUM(I,J,K), P(I,J,K));

AI.3 Defining Subscripts
In setting up a set of equations and variables use the following principles. Define a subscript for each physical phenomena set which has multiple members, i.e.,
Let
- $i$ denote production processes of which there are $I$
- $j$ denote locations of which there are $J$
- $k$ denote products of which there are $K$
- $m$ denote sales locations of which there are $M$.

Equivalent GAMS commands are
SET I /1*20/  
J /1*30/  
K /1*5/  
M /CHICAGO, BOSTON/;
Define different subscripts when you are either considering subsets of the subscript set or different physical phenomena.

AI.4 Defining and Using Variables
1. Define a unique symbol with a subscript for each manipulatable item.

For example:
\[ p_{ijk} = \text{production using process } i \text{ at location } j \text{ while producing good } k. \]
Or, equivalently,
PARAMETER P(I,J,K)

AI-6
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PARAMETER PRODUCTION(PROCESS, LOCATION, GOOD)
Here, for documentation purposes, the second expression is preferred.

2. Make sure that variable has the same subscript in each place it occurs.
Thus it is proper to write
\[
\text{Max } \sum_i \sum_j \sum_k t_{ijk}
\]
\[
\sum_i \sum_j t_{ijk} = 3 \quad \text{for all } k
\]
but wrong to write
\[
t_{ijk} \geq 0.
\]
The second model would cause error message $148$ indicating "DIMENSION DIFFERENT."

3. The authors feel it is a bad practice to define different items with the same symbol but varying subscripts. We think you should never use the same symbol for two different items as follows
\[
u_{ij} = \text{amount of tires transported from } i \text{ to } j \text{ and}
\]
\[
u_{kj} = \text{amount of chickens transported from } k \text{ to } j.
\]
GAMS would not permit this, giving error $150$ "Symbolic Equations Redefined."

Al.5 Equations
Modelers should carefully identify the conditions under which each equation exists and use subscripts to identify those conditions. We do not think modelers should try to overly compact the families of equations. For example, it is OK to define
\[
\sum_j a_{ij} x_j \leq b_i
\]
for all i, where \(a_{ij}\) is use of water by period and labor by period, where i denotes water periods and labor periods and \(b_i\) simultaneously contains water and labor availability by period. But we find it is better to define
\[
\sum_j d_{ij} x_j \leq e_i
\]
\[
\sum_j f_{ij} x_j \leq h_i
\]
where i denotes period,
d_i denotes water use and e_i water availability,
f_i denotes labor use and h_i labor availability.

AI.6 Cautions and Extensions

1. Be careful when you sum over terms which do not contain the subscript you are summing over. This is equivalent to multiplying a term by the number of items in the sum.

\[ \sum_{i=1}^{N} x_i = N x_i \]

\[ \sum_{j=1}^{3} x_j = 3(2) = 6 \]

Or, in GAMS

\texttt{SUM5A = SUM(J, X("2"));}

2. Be careful when you have a term in a family of equations which is of a lesser dimension than the family, this term will occur in each equation. For example, the expression

\[ \sum_{j} x_j = z_i \quad \text{for } i = 1, 2, 3 \]

implies that simultaneously

\[ \sum_{j} x_j = z_1 \]

\[ \sum_{j} x_j = z_2 \]

\[ \sum_{j} x_j = z_3. \]

3. The same rules as outlined above apply to product cases

\[ \prod_{i=1}^{3} x_i = x_1 x_2 x_3, \]

, equivalently,

\[ \text{PRODUCTX = PROD(I, X(I));} \]

4. The following relationships also hold for summation

a. \[ \sum_{i} K x_i = K \sum_{i} x_i \]

AI-8

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b. \[ \sum_{i=1}^{n} KP = K \sum_{i=1}^{n} P = KnP \]

c. \[ \sum_{i} \sum_{j} (v_{ij} + y_{ij}) = \sum_{i} \sum_{j} v_{ij} + \sum_{i} \sum_{j} y_{ij} \]

d. \[ \sum_{i} \sum_{j} (x_{ij} + y_{ij}) = n \sum_{i} x_{ij} + \sum_{i} \sum_{j} y_{ij} \text{ when } j = 1,2,...,n \]
### Table 1. Sample GAMS Commands for Summation Notation Expressions

```plaintext
1  **********************************************************************
2  ** THIS FILE CONTAINS GAMS EXAMPLES IN SUPPORT **
3  ** OF THE NOTES USING THE SUMMATION NOTATION **
4  **********************************************************************
5
6  SETS
7      I /1*3/  
8      J /1*2/  
9      K /1*2/  
10     M /1*2/  
11     N /1*3/  
12
13  PARAMETERS
14
15     X(I) /1 1,2 2,3 3/  
16     Z(I) /1 2,2 4,3 6/  
17
18  TABLE Y(I,J)
19
20        1  2  
21      1  2  3  
22      2  4  1  
23      3  1  4;
24
25  TABLE V(I,J)
26
27        1  2  
28      1  2  3  
29      2  4  1  
30      3  1  4;
31
32  TABLE P(I, J, K)
33
34       1.1  1.2  2.1  2.2  
35       1  1  3  5  7  
36       2  2  4  6  8  
37       3  1  2  3  4 ;
38
39  TABLE Q(M, N)
40
41       1  2  3  
42       1  5  10  
43       2  10  5  1;
44
45  **********************************
46  ** AI.1.1 SUM OF AN ITEM **
47  **********************************
48
49  PARAMETER
50    SUM1    SUM OF AN ITEM;  
51    SUM1 = SUM(I, X(I));
```

**AI-10**

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** AI.1.2 MULTIPLE SUMS **

```gams
PARAMETER
SUM2 MULTIPLE SUMS;
SUM2 = SUM((I,J), Y(I,J));
DISPLAY SUM2;
```

** Table 1. Sample GAMS Commands for Summation Notation Expressions (continued) **

** AI.1.3 SUM OF TWO ITEMS **

```gams
PARAMETERS
SUM3A SUM OF TWO ITEMS-1
SUM3B SUM OF TWO ITEMS-1
A(I) SUM OF TWO ITEMS-2
SUM4A SUM OF TWO ITEMS-3
SUM4B SUM OF TWO ITEMS-3;
SUM3A = SUM(I, X(I)+SUM(J, Y(I,J)));
SUM3B = SUM(I, X(I)) + SUM((I,J), Y(I,J));
A(I) = X(I) + SUM(J, Y(I,J));
SUM4A = SUM(I, X(I)+Z(I));
SUM4B = SUM(I, X(I)) + SUM(I, Z(I));
DISPLAY SUM3A, SUM3B, A, SUM4A, SUM4B;
```

** AI.2.1 FOR A SCALER EQUATION **

```gams
PARAMETERS
B1 SUM FOR A SCALER EQUATION-1;
B1 = SUM((I,J,K), P(I,J,K)) + SUM((M,N), Q(M,N));
DISPLAY B1;
```

* $OFFTEXT

** AI.2.2 FOR A FAMILY OF EQUATIONS **

```gams
B2 SUM FOR A SCALER EQUATION-2
B3 SUM FOR A SCALER EQUATION-3;
B2 = P(I,J,K) + Q(M,N);
B3 = SUM((I,J), P(I,J,K)) + SUM((M,N), Q(M,N));
DISPLAY B2, B3;
```

* $OFFTEXT

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VARIABLES  C(I), D(I,J), E1(I,J), F(J);

EQUATIONS
EQC(I) EQUATION C
EQD(I,J) EQUATION D
EQE1(I,J) EQUATION E1;

EQC(I)..  C(I) =E= 2;
EQD(I,J)..  D(I,J) =E= 2;
EQE1(I,J)..  E1(I,J) =E= F(J);

* $ONTEXT
* THE FOLLOWING EXPRESSION IS INCORRECT
* ERROR MESSAGES WILL BE ENCOUNTERED
* VARIABLES E2(I,J);

Table 1. Sample GAMS Commands for Summation Notation Expressions (continued)

EQE2 EQUATION E2;
EQE2..  E2(I,J) =E= 2;

VARIABLES G1(I), H1(I,J);
EQUATIONS
EQG1(I) EQUATION G1
EQH1(I,J) EQUATION H1;

EQG1(I)..  G1(I) =E= SUM((J,K), P(I,J,K));
EQH1(I,J)..  H1(I,J) =E= SUM(K, P(I,J,K));

* $ONTEXT
* THE FOLLOWING EXPRESSIONS ARE INCORRECT
* ERROR MESSAGES WILL BE ENCOUNTERED
* VARIABLES G2(I), H2(I);
EQUATIONS
EQG2(I) EQUATION G2
EQH2(I) EQUATION H2;

EQG2(I)..  G2(I) =E= P(I,J,K);
EQH2(I)..  H2(I) =E= SUM(K, P(I,J,K));

* $OFFTEXT

VARIABLES L1(I), U(I,M), R(I,J,K,M), S(I,J,M);
EQUATIONS
EQL1(I) EQUATION L1
EQN(I,M) EQUATION N;

EQL1(I)..  L1(I) =E= SUM((J,K), P(I,J,K));
EQN(I,M)..  U(I,M) =E= SUM((J,K), R(I,J,K,M)) + SUM(J, S(I,J,M));

* $ONTEXT
* THE FOLLOWING EXPRESSIONS ARE INCORRECT
* ERROR MESSAGES WILL BE ENCOUNTERED
* VARIABLES L2;
EQUATIONS
EQL2(I) EQUATION L2;

EQL2(I)..  L2 =E= P(I,J,K);

* OFFTEXT

VARIABLE O(I,M);

AI-12

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Table 1. Sample GAMS Commands for Summation Notation Expressions (continued)

172 ***************************************************************
173 ** AI.4 DEFINING AND USING VARIABLES **
174 ***************************************************************
175 VARIABLES
176 OBJ1 OBJECTIVE FUNCTION VALUE
177 T(I,J,K) DECISION VARIABLE;
178 EQUATIONS
179 OBJFUNC1 OBJECTIVE FUNCTION
180 CONST(K) CONSTRAINT;
181 OBJFUNC1.. OBJ1 =E= SUM((I,J,K), T(I,J,K));
182 CONST(K).. SUM((I,J), T(I,J,K)) =E= 3;
183 MODEL EXAMPLE1 /ALL/;
184 SOLVE EXAMPLE1 USING LP MAXIMIZING OBJ1;
185 DISPLAY T.L;
186 *
187 * $ONTEXT
188 * THE FOLLOWING COMMANDS ARE INCORRECT
189 * THEY WILL RESULT IN ERROR MESSAGES
190 VARIABLES
191 OBJ2 OBJECTIVE FUNCTION VALUE
192 TT(I,J,K) DECISION VARIABLE;
193 EQUATIONS
194 OBJFUNC2 OBJECTIVE FUNCTION
195 CONST(K) CONSTRAINT;
196 OBJFUNC2.. OBJ2 =E= SUM((I,J), TT(I,J));
197 CONST(K).. SUM((I,J), TT(I,J,K)) =E= 3;
198 MODEL EXAMPLE2 /ALL/;
199 SOLVE EXAMPLE2 USING LP MAXIMIZING OBJ2;
200 DISPLAY TT.L;
201 *
202 * $OFFTEXT
203 *
204 ***************************************************************
205 ** AI.6 CAUTIONS AND EXTENSIONS **
206 ***************************************************************
207 PARAMETER
PARAMETER

PRODUCT6 CAUTIONS AND EXTENSIONS-2;

PRODUCT6 = PROD(I, X(I));

DISPLAY PRODUCT6;

PARAMETERS

SUM7A CAUTIONS AND EXTENSIONS-3
SUM7B CAUTIONS AND EXTENSIONS-3
SUM8A CAUTIONS AND EXTENSIONS-4
SUM8B CAUTIONS AND EXTENSIONS-4
SUM8C CAUTIONS AND EXTENSIONS-4
SUM9A CAUTIONS AND EXTENSIONS-5
SUM9B CAUTIONS AND EXTENSIONS-5

Table 1. Sample GAMS Commands for Summation Notation Expressions (continued)

SUM10A CAUTIONS AND EXTENSIONS-6
SUM10B CAUTIONS AND EXTENSIONS-6;
SUM7A = SUM(I, 5*X(I));
SUM7B = 5*SUM(I, X(I));
SUM8A = SUM(I, 5*10);
SUM8B = 5*SUM(I, 10);
SUM8C = 5*3*10;
SUM9A = SUM((I,J), V(I,J)+Y(I,J));
SUM9B = SUM((I,J), V(I,J)) + SUM((I,J), Y(I,J));
SUM10A = SUM((I,J), X(I)+Y(I,J));
SUM10B = 2*SUM(I, X(I)) + SUM((I,J), Y(I,J));
DISPLAY SUM7A, SUM7B, SUM8A, SUM8B, SUM8C,
SUM9A, SUM9B, SUM10A, SUM10B;