

Two-Stage Buy-It-Now Auctions

Yu Yvette Zhang*, Rajiv Sarin †

Abstract

This paper studies auctions with a temporary Buy-It-Now (BIN) price using a two-stage model, in which two groups of bidders enter the auction at different times. The *early* bidders are offered a “Buy-It-Now” (BIN) option to purchase the item immediately at a listed price (BIN price). If no early bidder accepts the BIN option, an additional group of bidders (*late* bidders) enter the auction and both groups of bidders participate in a second-price sealed-bid auction without BIN option. When bidders have concave utility functions, we establish the existence and uniqueness of a cutoff equilibrium such that an early bidder will accept the BIN option if his valuation is higher than the cutoff valuation. Moreover, bidders are more likely to accept the BIN option when fewer bidders are offered the BIN option. We show that when facing risk averse bidders, the seller can obtain higher expected revenue by setting an appropriate BIN price. Furthermore, the expected seller revenue decreases with the number of early bidders. Therefore, comparing an auction with BIN option available to a subset of bidders and an auction with BIN option available to all bidders, a bidder is more likely to accept the BIN price in the former resulting in a higher expected seller revenue. These results may help explain the popularity of temporary BIN auctions on eBay and the observed high acceptance frequencies of BIN prices in experimental and field studies.

*Corresponding author. Department of Agricultural Economics, Texas A&M University, 4228 TAMU, College Station, TX 77843. Email: yzhang@tamu.edu

†Department of Economics, University of Exeter. Email: R.Sarin@exeter.ac.uk.

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1 Introduction

In recent years, an auction format called “Buy-It-Now” (BIN) auction has appeared in many auction sites such as eBay, Yahoo and Ubid. In a BIN auction, the bidders are allowed to buy the items at a fixed “Buy-It-Now” (BIN) price set by the seller and end the auction immediately. One important feature of eBay’s “Buy-It-Now” auction is that the BIN price is temporary and disappears as soon as a bid is made at or above the reserve price.

As of April 2008, eBay’s “Buy-It-Now” business made up 42% of all goods sold on eBay (Business Week, 2008). It is growing at an annual 22% pace, the fastest among eBay’s shopping business. The increasing popularity of the BIN auctions raises several interesting questions for economists. Under what conditions would a buyer purchase the item at the BIN price? Under what conditions would a seller set a BIN price in an auction? How would a seller set the reserve price and the BIN price in a BIN auction? There are several theoretical studies attempting to answer these questions (e.g. Hidvegi et al. 2006; Reynolds and Wooders 2009). While most of the existing models of BIN auctions assume the BIN option is offered to all bidders in an auction, our model is motivated by the observation that in eBay auctions, the BIN price is observed only by a portion of bidders who arrive early. If some early bidder places a bid above the reserve price, the BIN price disappears and the bidders who come late do not have the BIN option.

We construct a two-stage temporary BIN auction model with two groups of bidders entering the auction in different stages. In the first stage, a group of bidders (*early* bidders) are offered a “Buy-It-Now” (BIN) option by the seller to purchase the item immediately at a listed price (BIN price). If no early bidder accepts the BIN price, the BIN option disappears and the second stage starts with an additional group of bidders (*late* bidders). Both early and later bidders then participate in a second-price sealed-bid auction with no BIN option. Hence the late bidders do not have any information about the BIN price.

While most theoretical models in BIN auctions assume that the bidder utility functions are in special forms such as linear functions and CARA utility functions, our theoretical results are derived with general concave utility functions. We establish the existence and

uniqueness of a cutoff equilibrium such that an early bidder will accept the BIN option if his valuation is higher than the cutoff valuation. We show that if the seller sets the BIN price low enough, an early bidder accepts the BIN price only if his valuation is higher than an equilibrium cutoff valuation. This equilibrium cutoff valuation exists as long as the bidders' utility function is monotonic increasing and is unique when bidders are risk averse or risk neutral. Other things being equal, the cutoff valuation decreases with bidders' degree of risk aversion and increases with the number of early bidders. Therefore, risk averse bidders in our model are more likely to accept the BIN options than in the model which assumes that BIN option is available to all bidders.

When bidders are risk averse, by setting an appropriate BIN price, the seller can obtain a higher expected revenue in a two-stage BIN auction than in a second-price sealed-bid auction with the same reserve price, and the expected revenue decreases with the number of early bidders. Therefore, compared to an auction with BIN option available to all bidders, our model results in a higher expected seller revenue. These results may help explain the popularity of temporary BIN auctions on online auction sites such as eBay and the observed high acceptance rates of BIN prices in experimental and field studies.

The rest of the paper is organized as follows. In section 2 we review the current literature related to BIN auctions and discuss the contributions of this paper relative to the existing literature. We present our model in section 3, in which we characterize the bidders' equilibrium strategies in section 3.1, investigate the factors which affect the equilibrium in section 3.2, illustrate the equilibrium analysis of the entire auction using a simple example in section 3.3, and investigate the seller's revenue and identify the conditions under which a BIN auction raises more seller revenue than a standard second-price sealed-bid auction in section 3.4. Section 4 contains the conclusion of this paper. All mathematical proofs are gathered in Appendix.

2 Literature Review

Several studies have investigated the BIN auctions theoretically. Budish and Takeyama (2001) show that a seller facing two risk-averse bidders may improve its expected profit by using an optimal permanent BIN price. Hidvegi et al. (2006) extend the model with an arbitrary number of bidders and continuous valuation distribution and showed that the seller receives higher expected revenue in an auction with a permanent BIN price than in a standard second-price auction, provided that buyers are risk averse.

Reynolds and Wooders (2005) characterize equilibrium bidding strategies for bidders with CARA utility function in both temporary and permanent BIN auctions. They show that for both auctions, when bidders are risk averse with CARA utility function, by setting an appropriate BIN price, a seller can raise higher revenue than in a second-price sealed-bid auction. Mathews and Katzman (2006) consider temporary BIN auctions with risk neutral bidders and a risk averse seller. They find that when seller is risk averse, setting a BIN price may result in a Pareto improvement compared to a sealed bid second price auction. Mathews (2004) studies the impact of time discounting on the BIN auctions. He finds that time discounting by either the seller or the bidders can lead to the seller choosing a BIN price which results in the option being exercised with positive probability.

All the papers mentioned above assume that all bidders enter the auction simultaneously and are all offered a BIN option. However, one important feature of a temporary BIN price auction is that, if anyone places a bid higher than the reserve price, the BIN option disappears. Therefore, all bidders entering afterwards will not be able to observe the BIN price and thus participate in a standard second price auction. This creates an information heterogeneity among the bidders such that early bidders have observed the BIN price, while late bidders have no information about the BIN price. Consequently, those early bidders who observe the BIN price will not bid above the BIN price hence their valuation follow a truncated distribution. On the other hand, the bid distribution for the late bidders still follow the original distribution. Therefore, it is important to consider the case when only a portion of bidders are offered the BIN option.

Gupta and Gallien (2005) formulate a model featuring time-sensitive bidders with uniform valuations and Poisson arrivals. They solved seller's utility maximization problem by simulation and showed that the permanent BIN price auction leads to higher seller utility than the temporary BIN price auction does. However, they assumed that the seller maximizes his utility only by setting an optimal BIN price, without considering the reserve price. While the BIN price sets the upper-bound for the winning bid, the reserve price sets the its lower-bound. In a standard English auction, a revenue maximizing seller always set a reserve price that exceeds his value (Myerson 1981). Therefore, it is important to consider the reserve price in analyzing the auction equilibrium.

Ivanova-Stenzel and Kroger (2008) consider an auction in which the seller offers the BIN price only to one bidder, say bidder one. If bidder one accepts the BIN price then the auction ends. Otherwise, a standard second-price sealed-bid auction with no reserve price starts with n bidders. They derive equilibrium strategy for bidder one but not for the seller. Furthermore, they assume that a zero reserve price when solving the equilibrium. Because online auctions are usually monitored by a large number of potential bidders and last for more than a week, it is rare that only one bidder has observed the BIN price and has the opportunity to purchase the item immediately. Therefore, to better understand online auctions with a temporary BIN option, it is important to consider a general model that allows an arbitrary number of bidders who are offered the BIN option.

There are also some empirical and experimental studies on BIN auctions. Shahriar and Wooders (2007) suggest that a suitably chosen BIN price raises seller revenue with risk averse bidders. However they report that the frequency of BIN price being accepted is higher than theoretical prediction under the assumption that all bidders are offered the BIN option (Reynolds and Wooders, 2009). Similar results are also found by Ivanova-Stenzel and Kroger (2008) and Grebe, Ivanova-Stenzel and Kroger (2006).

In this study, we propose a model that captures the information heterogeneity among bidders in a temporary BIN auction by assuming that the BIN option is only available to one group of bidders. While most theoretical models in BIN auctions assume that the bidder

utility functions are in special forms such as linear functions and CARA utility functions, we establish our theoretical results for general concave utility functions $u(x)$ with $u(0) = 0$, $u'(x) > 0$ and $u''(x) \leq 0$.

There are two groups of bidders in the model: early bidders and late bidders. Early bidders enter the auction in stage one and observe the BIN price. They decide simultaneously whether to accept the BIN price and win the item immediately or to reject it. If no early bidder has accepted the BIN price, the BIN option disappears and the stage two starts and late bidders enter the auction. In stage two auction, both early bidders and late bidders participate in a second-price sealed-bid auction. We solve bidder's equilibrium strategy in a model allowing for an arbitrary number of risk neutral or risk averse bidders and continuous valuation distribution. When bidders are risk averse, we show that with some appropriately chosen BIN price, the seller expected revenue is higher in a BIN auction than in a second-price sealed-bid auction with the same reserve price. Furthermore, the seller revenue decreases with the number of early bidders. These results may help explain the popularity of temporary BIN auctions in online auction sites such as eBay. They also provide an explanation of some experimental and field findings that the frequency of bidders accepting BIN price is often higher than the theoretical predictions under the assumption that all bidders are offered the BIN option. The model is described below.

3 The Model

We assume that there is one indivisible good in the auction. The seller sets a reserve price r and a BIN price p . There are n bidders with each bidder's valuation drawn from a common continuous distribution $F(v)$ with support $[\underline{v}, \bar{v}]$ and density $f(v) = F'(v)$. Bidders are risk averse or risk neutral with utility function $u(x)$. Assume $u(0) = 0$, $u'(x) > 0$ and $u''(x) \leq 0$.

The auction is assumed to consist of the following two stages.

Stage One There are n_1 ($0 < n_1 \leq n$) early bidders at the beginning of the auction. All early bidders observe the BIN price p and decide simultaneously whether to accept p or

not. If an early bidder accepts p , he wins the item immediately and the auction ends without entering the second stage. If there are more than one early bidder accept the BIN price, a winner will be selected randomly among them. If no one accepts p (i.e., all early bidders reject p), the BIN option disappears and auction enters the second stage.

Stage Two There are n_2 late bidders ($0 \leq n_2 < n$ and $n_1 + n_2 = n$) entering the auction at stage two with no information about the BIN price. Both early bidders and late bidders participate in a standard second-price sealed-bid auction with a reserve price r . The highest bidder among them wins the auction. The winning price is the second highest bid.

3.1 Bidders' Equilibrium Strategies

We first derive bidders' equilibrium strategy in the two-stage auction presented above. Reynolds and Wooders (2009) show that under CARA, there is no loss of generality in restricting attention to a "cutoff" strategy in equilibria of stage one. A cutoff strategy for an early bidder i in stage one is characterized by a value $c \in [p, \bar{v}]$ such that he accepts the BIN price p if $v_i \geq c$ and reject the BIN price p if $v_i < c$.

Suppose one early bidder accepts the BIN price. The probability that there are another $k_1 - 1$ early bidders ($1 \leq k_1 \leq n_1$) who accept the BIN price is

$$\binom{n_1 - 1}{k_1 - 1} (1 - F(c))^{k_1 - 1} F^{n_1 - k_1}(c).$$

Define the expected payoff of an early bidder with valuation v_i who accepts the BIN price as $\pi_p^1(v_i)$,

$$\pi_p^1(v_i) = \sum_{k_1=1}^{n_1} \binom{n_1 - 1}{k_1 - 1} (1 - F(c))^{k_1 - 1} F^{n_1 - k_1}(c) \frac{u(v_i - p)}{k_1} = u(v_i - p) \frac{1 - F^{n_1}(c)}{n_1(1 - F(c))}.$$

When a bidder rejects the BIN price, he will not obtain the item if any other early bidders

accept the BIN price. If no bidder in stage one accepts the BIN price, all early bidders participate in a standard second-price sealed-bid auction in stage two with the addition of n_2 late bidders.

The auction enters the stage two with a probability of $F^{n_1}(c)$ when no early bidder accepts the BIN price. Since an early bidder rejects the BIN price only if his valuation is lower than the cutoff point c , the valuations of early bidders in stage two follows a truncated distribution $F_e(v)$ with support $[\underline{v}, c]$. Define the truncated distribution

$$F_e(v) = \begin{cases} F(v)/F(c), & \text{if } x \in [\underline{v}, c] \\ 1, & \text{otherwise} \end{cases}$$

with corresponding density function

$$f_e(v) = \begin{cases} f(v)/F(c), & \text{if } x \in [\underline{v}, c] \\ 0, & \text{otherwise} \end{cases}$$

In a second-price sealed-bid auction, it is a weakly dominant strategy for a bidder to bid his true valuation (Vickrey, 1961), which implies the following result.

Lemma 1 *If the auction enters stage two where a second-price sealed-bid auction occurs, both early bidders and late bidders bid their true valuations.*

According to Lemma 1, an early bidder wins the auction in stage two if: (a) all $n_1 - 1$ other early bidder also reject the BIN option in stage one; (b) his valuation is higher than all $n_1 - 1$ other early bidders and all n_2 late bidders. Define the expected payoff in stage two for an early bidder with valuation v_i who rejects the BIN price in stage one as $\pi_b^1(v_i)$,

$$\pi_b^1(v_i) = F^{n_1-1}(c)[u(v_i - r)(F_e^{n_1-1}(r)F^{n_2}(r)) + \int_r^{v_i} u(v_i - x)dF_e^{n_1-1}(x)F^{n_2}(x)].$$

Having obtained the expected payoff for an early bidder of accepting a BIN price and rejecting a BIN price, we proceed to derive an equilibrium cutoff valuation. A cutoff c^* is a symmetric Bayes-Nash equilibrium if $\pi_p^1(v_i) \leq \pi_b^1(v_i)$ for all $v_i \in [\underline{v}, c^*]$ and $\pi_b^1(v_i) < \pi_p^1(v_i)$

for all $v_i \in (c^*, \bar{v}]$. By the continuity of the payoff function, for the conjectured strategy to characterize an equilibrium in stage one, an early bidder's expected payoff from accepting the BIN price should be equal to his expected gain in stage two when his valuation equals c . Therefore, an equilibrium cutoff c^* satisfies $\pi_p^1(c^*) = \pi_b^1(c^*)$. Since

$$\begin{aligned}\pi_p^1(c) &= u(c-p) \frac{1-F^{n_1}(c)}{n_1(1-F(c))} \\ \pi_b^1(c) &= F^{n_1-1}(c) [u(c-r) F_e^{n_1-1}(r) F^{n_2}(r) + \int_r^c u(c-x) dF_e^{n_1-1}(x) F^{n_2}(x)] \\ &= F^{n_1-1}(c) [u(c-r) \frac{F^{n_1-1}(r)}{F^{n_1-1}(c)} F^{n_2}(r) + \int_r^c u(c-x) dF_e^{n_1-1}(x) F^{n_2}(x)] \\ &= u(c-r) F^{n-1}(r) + \int_r^c u(c-x) dF^{n-1}(x)\end{aligned}$$

it follows that

$$u(c^*-p) \frac{1-F^{n_1}(c^*)}{n_1(1-F(c^*))} = u(c^*-r) F^{n-1}(r) + \int_r^{c^*} u(c^*-x) dF^{n-1}(x).$$

In Theorem 1 we will show the conditions for Equation ?? to define an unique equilibrium cutoff valuation implicitly.

Following Reynold and Wooders (2009), we apply the concept of certainty equivalent payment in the proof of Theorem 1. We denote a certainty equivalent payment by $\delta(v)$ such that a risk averse bidder with valuation $v \in [\underline{v}, \bar{v}]$ which follows a distribution function $F(v)$ is indifferent between the following two outcomes: (a) winning a standard second-price auction with n bidders thus paying a random amount of $\max\{r, y\}$ where r is the reserve price and y is the second highest bid; (b) winning the auction and paying a certain amount $\delta(v)$. Using the mean value theorem, one can show that there exists a unique $\delta(v)$ such that

$$u(v-\delta(v)) F^{n-1}(v) = u(v-r) F^{n-1}(r) + \int_r^v u(v-x) dF^{n-1}(x),$$

where $\delta(v)$ is increasing in v . In particular, we define the certainty equivalent payment of a risk neutral bidder as $\delta_0(v)$.

We establish the following result.

Theorem 1 *Suppose that all bidders are risk averse with a twice-differentiable utility function $u(x)$ such that $u(0) = 0$, $u'(x) > 0$ and $u''(x) \leq 0$. Consider a two-stage temporary Buy-It-Now auction with n_1 early bidders and n_2 late bidders whose valuation follow a distribution $F(v)$ with support $[\underline{v}, \bar{v}]$. Suppose that the seller sets the reserve price at r and sets a BIN price at p .*

1. *If $p \leq \delta(\bar{v})$, there exists a unique symmetric equilibrium cutoff $c^* \in [p, \bar{v})$ for early bidders such that an early bidder accepts the BIN price if his valuation is higher than or equal to c^* and rejects the BIN price otherwise.*

The equilibrium cutoff c^ is implicitly defined by*

$$u(c^* - p) \frac{1 - F^{n_1}(c^*)}{n_1(1 - F(c^*))} = u(c^* - r)F^{n-1}(r) + \int_r^{c^*} u(c^* - x)dF^{n-1}(x).$$

This equilibrium cutoff exists when $u(0) = 0$, $u'(x) > 0$ and is unique when $u''(x) \leq 0$.

2. *If $p > \delta(\bar{v})$, there does not exist an equilibrium cutoff strategy. Hence the BIN price is never accepted by a bidder. Therefore a second-price sealed-bid auction with n ($n = n_1 + n_2$) bidders always occur in stage two.*

Proof. *See Appendix. ■*

Theorem 1 establishes the existence and uniqueness of an equilibrium cutoff strategy for early bidders such that an early bidder accepts the BIN price if his valuation is higher than or equal to a cutoff valuation and reject it otherwise. Note that this cutoff equilibrium exists as long as $u(0) = 0$ and $u'(x) > 0$. Hence risk aversion of the bidders is a sufficient but not necessary condition for the existence of the cutoff equilibrium. The cutoff equilibrium also exists when bidders are risk averse. Therefore, as long as seller sets the BIN price low enough, a bidder, who can be either risk neutral, risk averse or risk loving, will accept the BIN price as long as his valuation is lower than the cutoff valuation. However, the uniqueness

of the equilibrium is established under the assumption that $u''(x) \leq 0$, i.e. bidders are risk averse or risk neutral.

3.2 Factors Influencing Equilibrium Cutoff Valuation

In this section we investigate how this equilibrium cutoff valuation is affected by factors such as the proportion of early bidders, the BIN price, and bidders' degree of risk aversion.

Proportion of early bidders We first consider the proportion of early bidders. Fixing the number of total bidders, when there are fewer early bidders in stage one, an early bidder with valuation higher than the cutoff valuation has a greater probability of winning the auction. Meanwhile, the number of total bidders in stage two is the same thus the winning probability for an early bidder in stage two does not change. Therefore, it is more likely for an early bidder to accept the BIN price with fewer early bidders. This result is established formally below.

Corollary 1 *Consider a temporary Buy-It-Now auction with n_1 early bidders and n_2 late bidders whose valuation follow a distribution $F(v)$ with support $[\underline{v}, \bar{v}]$. Assume bidders are risk averse. Suppose that the seller sets the reserve price at r and the BIN price at $p < \delta(\bar{v})$. Fixing the total number of bidders, the unique symmetric equilibrium cutoff c^* for the early bidders to accept the BIN price increases with the number of early bidders.*

Proof. As we have shown in Theorem 1, when $p < \delta(\bar{v})$, a unique $c^* \in [p, \bar{v}]$ exists such that

$$u(c^* - p) \frac{1 - F^{n_1}(c^*)}{n_1(1 - F(c^*))} = u(c^* - r)F^{n-1}(r) + \int_r^{c^*} u(c^* - x)dF^{n-1}(x).$$

Define $L(c) = u(c - p) \frac{1 - F^{n_1}(c)}{n_1(1 - F(c))}$ and $R(c) = u(c - r)F^{n-1}(r) + \int_r^c u(c - x)dF^{n-1}(x)$, then we

have

$$\begin{aligned}
L(p) &= 0 \\
R(p) &= u(p-r)F^{n-1}(r) + \int_r^p u(p-x)dF^{n-1}(x) \geq L(p) \\
L(\bar{v}) &= u(c-p) \\
R(\bar{v}) &= u(\bar{v}-\delta(\bar{v})).
\end{aligned}$$

The total number of the bidders n does not change, therefore, as n_1 increases, $R(c)$ remains the same. Also, both $L(p)$ and $L(\bar{v})$ are not affected by n_1 . We have

$$L(c) = u(c-p)Q(F(c)),$$

where $Q(F(c)) = \frac{1-F(v)^{n_1}}{n_1(1-F(v))}$, which decreases as n_1 increases for $c < \bar{v}$. Thus, for any $c \in (p, \bar{v})$, $L(c)$ decreases n_1 increases.

In all, when n_1 increases, $R(c)$ remains the same for $c \in [p, \bar{v}]$, $L(c)$ remains the same only when $c = p$ and $c = \bar{v}$, and $L(c)$ becomes smaller for $c \in (p, \bar{v})$. We have also shown that $R(p) \geq L(p)$, $R(\bar{v}) \geq L(\bar{v})$ and $\frac{\partial L(c)}{\partial c} > \frac{\partial R(c)}{\partial c}$. Therefore the intersection of $L(c)$ and $R(c)$ shifts rightwards as n_1 increases. This suggests that the unique equilibrium cutoff c^* satisfying $L(c) = R(c)$ increases as n_1 increases. ■

Suppose that $n_1 = n$ and $n_2 = 0$. This is equivalent to the case where all bidders are offered the BIN option. Corollary 1 implies that, since the cutoff valuation attain its maximum in this case, a bidder is less likely to accept a BIN price compared with the case where only a portion of bidders are offered the BIN option. This may explain the high frequency of BIN price being taken in the empirical studies of BIN auctions.

BIN Price We now consider the impact of the BIN price on the equilibrium cutoff valuation. We establish the following result.

Corollary 2 *Consider a temporary Buy-It-Now auction with n_1 early bidders and n_2 late*

bidders whose valuation follow a distribution $F(v)$ with support $[\underline{v}, \bar{v}]$. Assume bidders are risk averse. When a seller sets the reserve price at r and sets a BIN price at p such that $p < \delta(\bar{v})$, the unique symmetric equilibrium cutoff $c^* \in [p, \bar{v})$ increases with the BIN price p .

Proof. When $p < \delta(\bar{v})$, c^* satisfies $L(c^*) = R(c^*)$, where $L(c) = u(c - p) Q(F(c))$ and $R(c) = u(c - r)F^{n-1}(r) + \int_r^c u(c - x)dF^{n-1}(x)$.

In the proof of Theorem 1, we have shown that for $c \in [p, \bar{v}]$, $L'(c) > 0$ and $R'(c) > 0$, $L(p) < R(p)$ and $L(\bar{v}) > R(\bar{v})$ if $p < \delta(\bar{v})$. When p increases, $L(c)$ decreases thus shifts rightward and $R(c)$ remains the same. Therefore the intersection of $L(c)$ and $R(c)$ shifts rightward, which means c^* increases. ■

This result is intuitive in the sense that a buyer will prefer a lower BIN price than a higher BIN price.

Risk Attitude A BIN option offers the bidder an opportunity to win the item with high certainty. Therefore a more risk averse bidder should be more likely to accept the BIN option compared with a less risk averse bidder.

Corollary 3 Consider a two-stage temporary Buy-It-Now auction with n_1 early bidders and n_2 late bidders. When a seller sets the reserve price at r and sets a BIN price at p such that $p < \delta(\bar{v})$, the unique symmetric equilibrium cutoff $c^* \in [p, \bar{v})$ decreases as bidders become more risk averse, that is, as the certainty equivalent payment, $\delta(v)$, of all bidders increases.

Proof. Define

$$L(c) = u(c - p) \frac{1 - F^{n_1}(c)}{n_1(1 - F(c))} = u(c - p)Q(F(c))$$

$$R(c) = u(c - \delta(c))F^{n-1}(c)$$

For utility function $u_k(v)$, we define the certainty equivalence payment $\delta_k(v)$ as

$$u_k(v - \delta_k(v))F^{n-1}(v) = u(v - r)F^{n-1}(r) + \int_r^v u(v - x)dF^{n-1}(x)$$

$$u_k(v - \delta_k(v)) = \frac{u(v - r)F^{n-1}(r) + \int_r^v u(v - x)dF^{n-1}(x)}{F^{n-1}(v)}$$

and let

$$L_k(c) = u_k(c - p) \frac{1 - F^{n_1}(c)}{n_1(1 - F(c))} = u_k(c - p)Q(F(c))$$

$$R_k(c) = u_k(c - \delta_k(c))F^{n_1-1}(c).$$

Denote c_k^* as the solution for $L_k(c_k^*) = R_k(c_k^*)$, which means,

$$u_k(c_k^* - p)Q(F(c_k^*)) = u_k(c_k^* - \delta_k(c_k^*))F^{n_1-1}(c_k^*).$$

For the same v , suppose $\delta_1(v) < \delta_2(v)$ for different utility function u_1 and u_2 . To show that c_k^* decreases in δ_k , it is sufficient to show that $c_1^* > c_2^*$.

As shown in the proof of Theorem One, we have shown that for $L(c) < R(c)$ if $p < c < c^*$, $L(c) = R(c)$ if $c = c^*$ and $L(c) > R(c)$ if $c^* < c < \bar{v}$. Therefore, to show that $c_1^* > c_2^*$, it is sufficient to show that $L_1(c_2^*) < R_1(c_2^*)$, or $L_2(c_1^*) > R_2(c_1^*)$.

Because $p < \delta_1(c_1^*) < \delta_2(c_1^*)$, we have $u_2(c_1^* - p) > u_2(c_1^* - \delta_2(c_1^*))$. And since $Q(F(c_1^*)) > F^{n_1-1}(c_1^*)$, we have

$$u_2(c_1^* - p)Q(F(c_1^*)) > u_2(c_1^* - \delta_2(c_1^*))F^{n_1-1}(c_1^*),$$

that is $L_2(c_1^*) > R_2(c_1^*)$.

Hence, we have established that $c_1^* > c_2^*$. Therefore, when $p < \delta(\bar{v})$, the unique symmetric equilibrium cutoff $c^* \in [p, \bar{v}]$ decreases as a bidder's certainty equivalent payment $\delta(v)$ increases, that is, as the bidder's utility function becomes more risk averse. ■

So far we have identified the equilibrium strategies for both early bidders and late bidders given the reserve price r and the BIN price p . To characterize the equilibrium for the entire auction, the seller's choice of reserve price and the BIN price must also be analyzed. We first assume that the seller is risk neutral and attaches no value to the good. The seller maximizes his revenue by setting a reserve price r and a BIN price p assuming that the bidders follow the equilibrium strategies characterized in Lemma 1 and Theorem 1. Below we illustrate the

equilibrium analysis of a temporary two-stage BIN auction two risk neutral bidders.

3.3 Equilibrium Analysis of a BIN auction with one Early Bidder and one Late Bidder

Suppose there is one risk neutral early bidder with valuation v_1 and one risk neutral late bidder with valuation v_2 , whose valuations follow a uniform distribution with support $[0, b]$. A seller sets the reserve price r and BIN price p at the beginning of the auction. Assume one early bidder enters the auction at stage one. He faces two choices: 1) to accept the BIN price p and win the item immediately, or 2) to reject a bid then participate in a second price sealed-bid auction to compete with one late bidder.

Suppose the early bidder uses a cutoff strategy c that he accepts the BIN price if $v_1 \geq c$ and rejects the BIN price otherwise.

Define the expected utility of an early bidder from accepting the BIN price as π_p^1 ,

$$\pi_p^1 = u(v_1 - p).$$

If the early bidder does not accept the BIN price p , the auction becomes a standard second price auction with two bidders. Define the early bidder's expected payoff in the auction as π_b^1 ,

$$\pi_b^1 = u(v_1 - r)F(r) + \int_r^{v_1} u(v_1 - x)dF(x).$$

If the early bidder's valuation exceeds his cutoff point, he will purchase the item at the BIN price in stage one and the auction ends. Therefore, if an early bidder shows up in stage two, his valuation must be lower than his cutoff point c . A later bidder obtains the item if his valuation is higher than the reserve price and the early bidder's valuation. Therefore, the late bidder's expected payoff, π^2 , is

$$\pi^2 = \frac{1}{F(c)}(u(v_2 - r)F(r) + \int_r^{v_2} u(v_2 - x)dF(x)).$$

By the continuity of the payoff function, for the cutoff strategy to characterize an equilibrium in stage one, an early bidder's expected payoff by taking the BIN price in stage one should be equal to his expected gain in stage two when his valuation equals to the equilibrium cutoff c^* , that is,

$$u(c^* - p) = u(c^* - r)F(r) + \int_r^{c^*} u(c^* - x)dF(x) \quad (1)$$

Since $F(v) = \frac{v}{b}$, and bidders are risk neutral equation 1 becomes,

$$c^* - p = (c^* - r)\left(\frac{r}{b}\right) + \int_r^{c^*} (c^* - x)d\frac{x}{b}.$$

The above equation have two roots, c_l and c_h ,

$$\begin{aligned} c_l^* &= b - \sqrt{b^2 - 2bp + r^2} \\ c_h^* &= b + \sqrt{b^2 - 2bp + r^2}. \end{aligned}$$

The higher root exceeds the highest bidder's valuation b , therefore there exists one unique cutoff point $c^* = b - \sqrt{b^2 - 2bp + r^2} \in [0, b]$ when $p \leq \frac{b^2+r^2}{2b}$.

Given the characterized bidder's equilibrium cutoff strategy, a seller maximize his expected revenue by choosing the reserve price r and the BIN price p . A seller's expected revenue from a stage one bidder is

$$\int_r^{c_1} \left\{ \int_0^r r dF(y) + \int_r^x y dF(y) \right\} dF(x) + \int_{c_1}^b p dF(x),$$

and the expected revenue from a stage two bidder is

$$\int_r^b \left(\frac{1}{F(c)} \int_r^y x dF(x) dF(y) \right).$$

Therefore, a seller's expected revenue is,

$$\begin{aligned}
\Pi &= \int_r^{c_1} \left\{ \int_0^r r dF(y) + \int_r^x y dF(y) \right\} dF(x) + \int_{c_1}^b p dF(x) + \int_r^b \left(\frac{1}{F(c)} \int_r^y x dF(x) dF(y) \right) \\
&= \int_r^c \left\{ \int_0^r r d\left(\frac{y}{b}\right) + \int_r^x y d\left(\frac{y}{b}\right) \right\} d\left(\frac{x}{b}\right) + \int_{c_1}^b p d\left(\frac{x}{b}\right) + \int_r^b \left(\frac{1}{F(c)} \right) \int_r^y x d\left(\frac{x}{b}\right) d\left(\frac{y}{b}\right) \\
&= \frac{1}{b^2} \left(\frac{c^3 - r^3}{3} + \frac{c^2 - r^2}{2} (b - c) \right) + \frac{(b - c)}{b} p.
\end{aligned}$$

Assuming the early bidder is using the cutoff point $c^* = b - \sqrt{b^2 - 2bp + r^2}$, we have,

$$\begin{aligned}
\Pi &= \frac{1}{b^2} \left(\frac{(b - \sqrt{b^2 - 2bp + r^2})^3 - r^3}{3} + \frac{(b - \sqrt{b^2 - 2bp + r^2})^2 - r^2}{2} \sqrt{b^2 - 2bp + r^2} \right) \\
&\quad + \frac{\sqrt{b^2 - 2bp + r^2}}{b} p.
\end{aligned}$$

The optimal reserve price r^* and the optimal BIN price p^* can be obtained by solving $\partial_1 \Pi(r, p) = 0$ and $\partial_2 \Pi(r, p) = 0$. Solve for $\partial_1 \Pi(r, p) = 0$, we have

$$r^* = 0$$

and $SOC = -\frac{1}{b} < 0$.

Solve for $\frac{\partial \Pi_c(r, p)}{\partial p} = 0$, we have

$$p^* = \frac{r^2 + b^2}{2b},$$

$SOC < 0$ when $p < \frac{b}{2}$ and SOC goes to infinity as $p \geq \frac{b}{2}$. Therefore

$$p^* \geq \frac{b}{2}.$$

Perviously, we found that in order for the cutoff point to exist, we need $p \leq \frac{b^2 + r^2}{2b}$. Hence the cutoff equilibrium exists when $p^* = \frac{b}{2}$ and does not exist when $p^* > \frac{b}{2}$. Therefore the optimal seller strategy given the bidder uses a cutoff strategy is $p^* = \frac{b}{2}$ and $r^* = 0$. Then we have

$$c^* = b,$$

which means bidder one will only accept the BIN price when his valuation is b .

If a seller sets the BIN price p^* at $\frac{b}{2}$ and the reserve price r at 0, the cutoff valuation for an early bidder to accept the BIN price is $c^* = b$ and the seller's expected revenue is

$$\Pi_1^* = \frac{1}{b^2} \left(\frac{c^{*3} - r^3}{3} + \frac{c^{*2} - r^2}{2} (b - c_{1l}^*) \right) + \frac{(b - c^*)}{b} p = \frac{b}{3}.$$

If a seller sets the BIN price $p^* > \frac{b}{2}$, then the cutoff equilibrium does not exist, which suggests that the early bidder will never accept the BIN price. In this case, a standard second-price sealed-bid auction with two bidders always occur. Thus a seller's expected revenue becomes

$$\begin{aligned} \Pi &= 2 \int_r^b [rF(r) + \int_r^{v_i} x dF(x)] dF(v_i) \\ &= 2r[1 - F(r)]F(r) + \int_r^b 2x[1 - F(x)]f(x)dx. \end{aligned} \quad (2)$$

It follows that

$$\begin{aligned} \partial\Pi/\partial r &= 2F(r)(1 - F(r) - rf(r)) \\ \partial\Pi/\partial r &= 0 \text{ when } r^* = \frac{b}{2}, \text{ and } SOC < 0. \end{aligned}$$

Plug $r^* = \frac{b}{2}$ into equation 2, we obtain the expected seller revenue, Π_2^* , when he sets a BIN price higher than $\frac{b}{2}$,

$$\Pi_2^* = \frac{5}{12}b,$$

which is higher than the revenue when he sets the BIN price at $\frac{b}{2}$.

Therefore, in a temporary Buy-It-Now auction with one risk neutral early bidder and one risk neutral late bidder, a risk neutral seller will choose a reserve price $r = \frac{b}{2}$ and a BIN price $p > \frac{b}{2}$ such that bidder one will never exercise the buy-it-now option and a second-price sealed-bid auction always occur. The maximum seller revenue in this case is $\frac{5}{12}b$, which is the same as the the maximum seller revenue in a second-price sealed-bid auction with two

risk neutral bidders.

3.4 Seller Revenue

Now we investigate the seller revenue assuming arbitrary numbers of bidders. Define a seller's expected payment from an early bidder with valuation v_i if the bidder accepts the buy price as $\Pi_{1,1}(r, p)$,

$$\Pi_{1,1}(r, p) = \sum_{k_1=1}^{n_1} \binom{n_1-1}{k_1-1} (1-F(c))^{k_1-1} F^{n_1-k_1}(c) \frac{p}{k_1} = p \frac{1-F^{n_1}(c)}{n_1(1-F(c))}.$$

Define the seller's expected payment from an early bidder if the bidder rejects the BIN price and no other early bidders take the BIN option as $\Pi_{1,2}(r, p)$,

$$\Pi_{1,2}(r, p) = F^{n_1-1}(c) [r(F_e^{n_1-1}(r)F^{n_2}(r)) + \int_r^{v_i} x dF_e^{n_1-1}(x)F^{n_2}(x)].$$

On the other hand, a seller's expected revenue from a late bidder is

$$\Pi_2(r, p) = F^{n_1-1}(c) [r(F_e^{n_1}(r)F^{n_2-1}(r)) + \int_r^{v_i} x dF_e^{n_1}(x)F^{n_2-1}(x)].$$

Therefore, a seller's total expected revenue, $\Pi(r, p)$, is

$$\begin{aligned} \Pi(r, p) &= n_1 \int_c^{\bar{v}} \Pi_{1,1}(r, p) dF(v_i) + n_1 \int_r^c \Pi_{1,2}(r, p) dF(v_i) + n_2 \int_r^{\bar{v}} \Pi_2(r, p) dF(v_i) \\ &= n_1 \int_c^{\bar{v}} p \frac{1-F^{n_1}(c)}{n_1(1-F(c))} dF(v_i) \\ &\quad + n_1 \int_r^c \{F^{n_1-1}(c) [r(F_e^{n_1-1}(r)F^{n_2}(r)) + \int_r^{v_i} x dF_e^{n_1-1}(x)F^{n_2}(x)]\} dF(v_i) \\ &\quad + n_2 \int_r^{\bar{v}} \{F^{n_1-1}(c) [r(F_e^{n_1}(r)F^{n_2-1}(r)) + \int_r^{v_i} x dF_e^{n_1}(x)F^{n_2-1}(x)]\} dF(v_i). \end{aligned}$$

We now first derive seller's optimal strategy when assuming bidders are risk neutral.

Theorem 2 *Consider a two-stage Buy-It-Now auction with n_1 early bidders and n_2 late bidders. Assume bidders are risk neutral and their valuation follow a distribution $F(v)$*

under the support of $[\underline{v}, \bar{v}]$. It is a weakly dominant strategy for a risk neutral seller to set a BIN price $p > \delta_0(\bar{v})$ such that the early bidders never accepts the BIN price, and a reserve $r = \frac{1-F(r)}{f(r)}$, which is the same optimal reserve price in a second-price sealed-bid auction. Thus the maximum revenue a risk neutral seller gains in a Buy-It-Now auction facing n_1 risk neutral bidders and n_2 risk neutral bidders is the same as that in a second-price sealed-bid auction with n bidders where $n = n_1 + n_2$.

Proof. When bidders are risk neutral, a second-price sealed-bid auction with the optimal reserve price is revenue maximizing (Myerson 1981). Therefore, it is a weakly dominant strategy for the seller set the BIN price $p > \delta(\bar{v})$ such that the early bidders will never accept the BIN price thus a second-price sealed-bid auction with n bidders will always occur.

In a second-price auction with n bidders, the expected seller revenue is

$$\begin{aligned}\Pi &= n \int_r^{\bar{v}} [rF^{n-1}(r) + \int_r^{v_i} x dF^{n-1}(x)] dF(v_i) \\ &= rn[1 - F(r)]F(r)^{n-1} + \int_r^{\bar{v}} xn(n-1)F(x)^{n-2}[1 - F(x)]f(x)dx.\end{aligned}$$

Differentiating Π with respect to r , we have

$$\begin{aligned}\partial\Pi/\partial r &= nF(r)^{n-1}(1 - F(r) - rf(r)) \\ \partial\Pi/\partial r &= 0 \text{ when } r^* = \frac{1 - F(r^*)}{f(r^*)}.\end{aligned}$$

Therefore, the seller's expected revenue is maximized by a unique r^* with $r^* = \frac{1-F(r^*)}{f(r^*)}$. ■

Theorem 2 implies that when bidders are risk neutral, to maximize his revenue, a seller should not set a BIN price, or should set a BIN price high enough, such that no bidder will exercise the “Buy-It-Now” option. This is because if the BIN price $p < \delta_0(\bar{v})$, there is a positive possibility that the BIN price will be accepted by some bidder thus the revenue for the seller is p . However, in a second-price auction, the bidder with highest valuation wins and pays $\delta_0(\bar{v})$, which is higher than p .

Up to now we have investigated the seller's choice in a two-stage Buy-It-Now auction

when bidders are risk neutral. We find that when bidders are risk neutral, the seller will set a BIN price high enough so that no bidders will accept the BIN price. Therefore a two-stage Buy-It-Now auction does not yield a higher seller revenue than a standard second-price sealed-bid auction. However, bidders are often risk averse. Bidders' risk attitudes do not affect seller revenue in a standard second-price sealed-bid auction (Milgrom and Weber, 1982). Meanwhile, as we have shown in Corollary 2, the more risk averse a bidder is, the more likely he is to accept a BIN price. The result below establishes conditions under which a two-staged buy-it-now auction raises higher expected seller revenue than a standard second-price sealed-bid auction, when a seller faces risk averse bidders.

Corollary 4 *Consider a temporary Buy-It-Now auction with n_1 early bidders and n_2 late bidders whose valuation follow a distribution $F(v)$ with support $[v, \bar{v}]$. If a seller sets the BIN price p such that $\delta_0(\bar{v}) < p < \delta(\bar{v})$, the two-stage BIN auction with n_1 early bidders and n_2 late bidders raises higher seller revenue compared with a second-price sealed-bid auction with the same reserve price r and n bidders ($n = n_1 + n_2$) when bidders are risk averse.*

Proof. As we have shown in Theorem 1, when $p < \delta(\bar{v})$, an early bidder with valuation $v_i > c^*$ will accept the BIN price, where c^* is implicitly given in theorem 1 and $\delta(v)$ is the certainty equivalent payment.

Without loss of generality, consider bidder one with valuation v_1 , which is the highest among all n_1 early bidders and bidder two with valuation v_2 , which is the highest among all n_2 late bidders.

(1) If $v_1 < c^*$, since bidder one has the highest valuation among all early bidders, then the BIN price is never accepted by any bidders. Hence a second-price sealed-bid auction starts in stage two with $n = n_1 + n_2$ bidders. In this case, the maximum seller revenue is the same as a second-price sealed-bid auction with n bidders. Therefore, if bidder one is the winner in stage two, then the seller's expected revenue is $\delta(v_1)$. If bidder one is not the winner in stage two, then bidder two must be the winner, therefore the seller's expected revenue is $\delta(v_2)$.

(2) If $v_1 > c^*$, then bidder one will accept the BIN price p if $p < \delta(\bar{v})$. Define $\delta_0(\bar{v})$ as the certainty equivalent payment for a risk neutral bidder with valuation \bar{v} . By the property of

the certainty equivalent payment, we have $\delta_0(v_1) < \delta_0(\bar{v})$, $\delta_0(v_2) < \delta_0(\bar{v})$ and $\delta_0(\bar{v}) < \delta(\bar{v})$. If a seller sets the BIN price p such that $\max\{\delta_0(v_1), \delta_0(v_2)\} < \delta_0(\bar{v}) < p < \delta(\bar{v})$, then bidder one will accept the BIN price. Hence the seller will be paid by p . Therefore the expected seller revenue when $v_1 > c^*$ is always higher than that when $v_1 < c^*$.

So we have shown that, when bidders are risk averse, compared with a second-price sealed-bid auction with reserve price r , if the seller sets the BIN price at $\delta_0(\bar{v}) < p < \delta(\bar{v})$ and the reserve price at r , the seller's expected revenue in a two-stage Buy-It-Now auction is (i) the same if $v_1 < c^*$; (ii) and higher if $v_1 > c^*$. Since $c^* \in [p, \bar{v})$, there is a positive possibility that $v_1 > c^*$. Therefore, the seller's *ex-ante* expected revenue is higher in the two-stage Buy-It-Now auction. ■

Note that as discussed in the proof of the Theorem 1, when bidders are risk loving, there is still a probability that the BIN price is accepted by a bidder as long as it is low enough. However, the certainty equivalent payment of a risk loving bidder is lower than a risk neutral bidder with the same valuation. Therefore, the expected seller revenue from a BIN auction is lower than that from a second-price sealed-bid auction when bidders are risk loving.

Next we show how seller's expected revenue is affected by bidders' risk attitude.

Corollary 5 *Suppose bidders are risk averse and their valuations follow a distribution function $F(v)$ with support $[\underline{v}, \bar{v}]$. If a seller sets the BIN price p such that $\delta_0(\bar{v}) < p < \delta(\bar{v})$, then the seller revenue in a two-stage BIN auction with n_1 early bidders and n_2 late bidders increases as bidders become more risk averse, that is, as the certainty equivalent payments of all bidders increase.*

Proof. Assume for the same v , $\delta_1(v) < \delta_2(v)$, for utility function u_1 and u_2 . Hence bidders with utility function u_1 are more risk averse than bidders with utility function u_2 . As shown in Corollary 2, the unique equilibrium cutoff c^* decreases as bidders become more risk averse, therefore $c_1^* > c_2^*$.

Without loss of generality, consider bidder one with valuation v_1 , which is the highest among all n_1 early bidders and bidder two with valuation v_2 , which is the highest among all

n_2 late bidders.

When bidders' utility function is u_2 , if $v_1 > c_1^*$, the BIN price will be accepted and the seller's expected revenue is p . If $v_1 < c_2^*$, the auction become second-price sealed-bid auction with $n_1 + n_2$ bidders, hence the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$. If $c_2^* < v_1 < c_1^*$, again no bidder will accept the BIN price and a second-price sealed-bid auction starts. thus the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$.

When bidders' utility function is u_1 , if $v_1 > c_1^*$, the BIN price will be accepted and the seller's expected revenue is p . If $v_1 < c_2^*$, the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$. However, if $c_2^* < v_1 < c_1^*$ bidder one will accept the BIN price thus seller's expected revenue is $p > \delta_0(\bar{v}) > \max\{\delta_0(v_1), \delta_0(v_2)\}$. Hence, when bidders' utility function is u_1 , the seller's ex-ante expected revenue is higher than the case when bidders' utility function is u_2 .

Therefore, we show that the seller revenue increases as bidders become more risk averse.

■

Below we establish that a BIN auction with less early bidders will generate higher seller revenue.

Corollary 6 *Suppose bidders are risk averse and their valuations follow a distribution function $F(v)$ with support $[\underline{v}, \bar{v}]$. Given a fixed total number of bidders n ($n = n_1 + n_2$), if a seller sets the BIN price p such that $\delta_0(\bar{v}) < p < \delta(\bar{v})$, then the seller revenue in a two-stage BIN auction with n_1 early bidders and n_2 late bidders increases as the proportion of the early bidders decreases.*

Proof. Fix the total number of bidders n . Let $0 < n_1 < n'_1 < n$. Define the cutoff valuation as c^* and c'^* when the number of early bidders is n_1 and n'_1 respectively. Proposition One shows that fixing the total number of bidders, the unique symmetric equilibrium cutoff c^* for the early bidders to accept the BIN price increases as n_1 increases. Therefore, $c^* < c'^*$.

Without loss of generality, consider bidder one with valuation v_1 , which is the highest among all n_1 early bidders and bidder two with valuation v_2 , which is the highest among all n_2 late bidders.

When there are n'_1 early bidders, if $v_1 > c'^*$, bidder one will accept the BIN price thus the seller's expected revenue is p . If $v_1 < c^* < c'^*$, then no bidder will accept the BIN price and the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$. If $c^* < v_1 < c'^*$, then again no bidder will accept the BIN price and a second-price sealed-bid auction starts. thus the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$.

When there are n_1 bidders, if $v_1 > c'^* > c^*$, then bidder one will accept the BIN price, then the seller's expected revenue is p . If $v_1 < c^*$, then no bidder will accept the BIN price and the seller's expected revenue is $\max\{\delta_0(v_1), \delta_0(v_2)\}$. If $c^* < v_1 < c'^*$, then bidder one will accept the BIN price, therefore the seller's expected revenue is $p > \max\{\delta_0(v_1), \delta_0(v_2)\}$.

Hence, for a fixed total number of bidders n , the seller's ex-ante expected revenue is higher when there are n_1 bidders compared with when there are $n'_1 > n_1$ bidders. Therefore, the seller's revenue increases as the number of the early bidders n_1 decreases. ■

Intuitively seller's expected revenue in a BIN auction increases with bidders' risk attitude: the more risk averse bidders are, the more likely they are to take the BIN option. On the other hand, the more early bidders in a two-stage BIN auction, the less likely is a bidder with valuation higher than the cutoff valuation to win the auction by accepting the BIN price. Thus the less likely he is to exercise the BIN option.

4 Conclusion

In this essay, we study auctions with a temporary Buy-it-now price using a two-stage model. We establish that when bidders are risk averse, if the seller sets the BIN price low enough, an early bidders will accept the BIN price only if his valuation is higher than a unique equilibrium cutoff valuation. Other things being equal, the cutoff valuation decreases with bidders' degree of risk aversion and increases with the proportion of early bidders. When bidders are risk neutral, we characterize the equilibrium for both seller and bidders. Facing risk neutral bidders, a seller will set the BIN high enough, such that no bidder will accept the BIN price thus starting a second-price sealed-bid auction. Therefore the seller revenue with risk neutral bidders in a BIN auction is the same as in a second-price sealed-bid auction.

This result is consistent with previous theoretical results under the assumption that the BIN price is available to all bidders (e.g. Hidvegi et al. 2006 and Reynolds and Wooders 2009).

We further show that when bidders are risk averse, by setting an appropriate BIN price, the seller obtains a higher expected revenue in a two-stage BIN auction than in a standard second-price sealed-bid auction, and the expected revenue decreases with the number of early bidders. Our result may help explain why empirical studies often report that the frequency of bidders accepting the BIN price is higher than theoretical predictions under the assumption that all bidders are offered the BIN option. Furthermore, seller revenue increases as the number of early bidders decreases. Therefore, by accounting for the fact that the BIN option is available only to one group of bidders, our model provides an explanation for the popularity of temporary BIN auctions in online auction sites such as eBay, where bidders enter the auction after the BIN option disappears do not have information about the BIN price.

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Appendix

Proof of Theorem 1. For a unique cutoff $c^* \in [p, \bar{v}]$ to exist such that an early bidder accepts the BIN price if his valuation $v_i > c$, rejects the BIN price if $v_i < c$ and indifferent between the two options if $v_i=c$, we need equation ?? to have one unique solution $c^* \in [p, \bar{v}]$.

Define $\gamma(x) = \frac{1-x^{n_1}}{n_1(1-x)} \leq 1$. It follows that $\gamma(1) = \lim_{x \rightarrow 1} \gamma(x) = 1$.

$$\gamma(x) = \frac{1-x^{n_1}}{n_1(1-x)} > \frac{1-x^n}{n(1-x)} > x^{n-1} \text{ for } x \in [0, 1) \text{ and } n > n_1.$$

Define $Q(v) = \gamma(F(v)) = \frac{1-F(v)^{n_1}}{n_1(1-F(v))}$ and $G(v) = F^n(v)$.

Then we have

$$Q(v) > G(v).$$

Define the left hand side of equation 1 as

$$L(c) = u(c-p)Q(c).$$

Apply the concept of the certainty equivalent payment, we define the right hand side as $R(c)$ and

$$R(c) = u(c-r)G(r) + \int_r^c u(c-x)dG(x) = u(c-\delta(c))G(c).$$

For a unique equilibrium cutoff $c^* \in [p, \bar{v}]$ to exist, $L(c)$ and $R(c)$ should have only one intersection when $p \leq c \leq \bar{v}$.

At equilibrium,

$$u(c-p)Q(c) = u(c-\delta(c))G(c)$$

Since $Q(c) > G(c)$ and $u(x)$ is concave, we have $u(c-p) < u(c-\delta(c))$ and $u'(c-p) > u'(c-\delta(c))$.

Plugging $c = p$ and \bar{v} into $L(c)$ and $R(c)$, we have

$$L(p) = 0;$$

$$R(p) = u(p-r)F^{n-1}(r) + \int_r^p u(p-x)dF^{n-1}(x) \geq L(p)$$

$$L(\bar{v}) = u(c-p)$$

$$R(\bar{v}) = u(\bar{v}-\delta(\bar{v}))$$

Lets consider the following cases.

Case 1: $p \leq \delta(\bar{v})$

We first consider the case when $p \leq \delta(\bar{v})$. Since $u'(x) > 0$, then $u(\bar{v} - p) > u(\bar{v} - \delta(\bar{v}))$. Hence, we have

$$L(p) \leq R(p) \text{ and } L(\bar{v}) > R(\bar{v}).$$

Therefore, the solution always exists as long as $u'(x) > 0$.

If $\frac{\partial L(c)}{\partial c} > \frac{\partial R(c)}{\partial c}$ when $c \in [p, \bar{v}]$, then $L(c)$ and $R(c)$ have one unique intersection in this domain.

$$\begin{aligned} \frac{\partial L(c)}{\partial c} &= u'(c-p)Q(c) + u(c-p)Q'(c) \\ \frac{\partial R(c)}{\partial c} &= u'(c-\delta(c))(1-\delta'(c))G(c) + u(c-\delta(c))G'(c) \\ &= u'(c-\delta(c))G(c) + u(c-\delta(c))G'(c) - u'(c-\delta(c))\delta'(c)G'(c) \end{aligned}$$

Since $u'(c-p) > u'(c-\delta(c))$ and $Q(c) > G(c)$, we have

$$u'(c-p)Q(c) > u'(c-\delta(c))G(c).$$

Hence $\frac{\partial L(c)}{\partial c} > \frac{\partial R(c)}{\partial c}$ if

$$\begin{aligned} u(c-\delta(c))G'(c) &\leq u'(c-\delta(c))\delta'(c)G'(c) \\ \delta'(c) &\geq \frac{u(c-\delta(c))G'(c)}{u'(c-\delta(c))G'(c)} \end{aligned}$$

By the defintion of the certainty equivalent payment,

$$u(c-\delta(c))G(c) = u(c-r)G(r) + \int_r^c u(c-x)dG(x).$$

Differentiate both sides, we obtain

$$u'(c - \delta(c))G(c) - u'(c - \delta(c))\delta'(c)G(c) + u(c - \delta(c))G'(c) = u'(c - r)G(r) + \int_r^c u'(c - x)dG(x).$$

Therefore,

$$\delta'(c) = \frac{u'(c - \delta(c))G(c) - u'(c - r)G(r) - \int_r^c u'(c - x)dG(x) + u(c - \delta(c))G'(c)}{u'(c - \delta(c))G(c)}.$$

Hence, $\delta'(c) \geq \frac{u(c - \delta(c))G'(c)}{u'(c - \delta(c))G(c)}$ if

$$u'(c - r)G(r) + \int_r^c u'(c - x)dG(x) \leq u'(c - \delta(c))G(c)$$

By mean value theorem, for any $x \in [r, \delta(c)]$, we can write

$$\begin{aligned} u'(c - x) &= u'(c - \delta(c) + \delta(c) - x) \\ &= u'(c - \delta(c)) + (\delta(c) - x)u''(y), \end{aligned}$$

where $y \in [c - \delta(c), c - x]$. Therefore,

$$\begin{aligned} &u'(c - r)G(r) + \int_r^c u'(c - x)dG(x) \\ &= u'(c - r)G(r) + \int_r^c (u'(c - \delta(c)) + (\delta(c) - x)u''(y))dG(x) \\ &= u'(c - r)G(r) + u'(c - \delta(c))(G(c) - G(r)) + \int_r^c (\delta(c) - x)u''(y)dG(x) \\ &= G(r)(u'(c - r) - u'(c - \delta(c))) + u'(c - \delta(c))G(c) + \int_r^c (\delta(c) - x)u''(y)dG(x) \\ &\leq u'(c - \delta(c))G(c) + \int_r^c (\delta(c) - x)u''(y)dG(x) \text{ since } r \leq \delta(c). \end{aligned}$$

Denote $M = \max u''(y)$. Since u is concave, $M < 0$. Therefore

$$\int_r^c (\delta(c) - x)u''(y)dG(x) \leq M \int_r^c (\delta(c) - x)dG(x) = M(\delta(c) - \int_r^c xdG(x)).$$

By the definition of $\delta(c)$ and the concavity of the utility function u , we have $\delta(c) \geq \int_r^c x dG(x)$ thus

$$\int_r^c (\delta(c) - x) u''(y) dG(x) \leq 0.$$

Hence, we obtain

$$u'(c - r)G(r) + \int_r^c u'(c - x) dG(x) \leq u'(c - \delta(c))G(c).$$

Therefore, we establish that

$$\frac{\partial L(c)}{\partial c} > \frac{\partial R(c)}{\partial c},$$

which, combining with $L(p) \leq R(p)$ and $L(\bar{v}) > R(\bar{v})$, implies the uniqueness and existence of the cutoff equilibrium equilibrium cutoff $c^* \in [p, \bar{v}]$.

Case 2: $p > \delta(\bar{v})$

Since $p > \delta(\bar{v})$, we have

$$u(\bar{v} - p) \leq u(\bar{v} - \delta(\bar{v})),$$

which means

$$L(\bar{v}) \leq R(\bar{v}).$$

In the proof of case 1 we have shown that $\frac{\partial L(c)}{\partial c} > \frac{\partial R(c)}{\partial c}$. Therefore, for one equilibrium $c^* \in [p, \bar{v}]$ to exist, we need $L(p) \geq R(p)$. However, $L(p) = 0$ and $R(p) \geq 0$. So the only possible solution is $c^* = \bar{v}$ when $p = \delta(\bar{v})$, otherwise $L(c) \leq R(c)$.

In conclusion, when the BIN price $p > \delta(\bar{v})$, an early bidder never accepts the BIN price in the equilibrium; when $p \leq \delta(\bar{v})$, there exists a unique symmetric equilibrium cutoff $c^* \in [p, \bar{v}]$ for early bidders such that an early bidder accepts the BIN price if his valuation is higher than c^* and rejects the BIN price otherwise. It follows that the equilibrium cutoff c^* is implicitly defined by

$$u(c^* - p) \frac{1 - F^{n_1}(c^*)}{n_1(1 - F(c^*))} = u(c^* - r) F^{n-1}(r) + \int_r^{c^*} u(c^* - x) dF^{n-1}(x),$$

which completes the proof of Theorem 1. ■